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Prepayment and the Valuation of Danish Mortgage-Backed Bonds

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Preface

The work on this thesis has benefited from the comments and participation of several people. I wish to thank Carsten Tanggaard, Esben Høg, Hans Jørgen Juhl and Anders Grosen for their advice and comments on many issues. The careful review by Peter Løchte Jørgensen of an early version has resulted in a large improvement of the grammatical level as well as the correction of several errors.

Various people at Nykredit, Kreditforeningen Danmark, BRF and Bikuben have kindly assisted in the collection of data. I also wish to acknowledge the helpful support from the students in charge of the Financial Database of the Aarhus School of Business.

I finally wish to thank Henrik Sørensen, GTJ FinansAnalyse, who has prepared, commented and summarized nearly all empirical simulations reported in this thesis. The work involved has been tremendous and without his support this thesis would not have been completed neither today nor this year.

Svend Jakobsen

1 Introduction and Summary

The purpose of the current thesis is to develop and test a pricing model for Danish callable[1] mortgage backed bonds (MBBs). MBBs account for more than 52% of the Danish bond market with a total outstanding face value of 652 billion DKK as of May 1992. MBBs are issued by mortgage credit institutions and each bond is backed by a pool of several thousands individual mortgages.

The pricing of MBBs is made complicated by the so-called prepayment option. The prepayment option allows each mortgage holder to prepay the loan at any time and change to a new loan at the current market rate of interest. As mortgage holders tend to prepay high coupon mortgages when market rates are low the prepayment option may severely affect the return to bond holders.

Table 1.1: *Bonds listed at the Copenhagen Stock Exchange*, *May 1992*[2].

Bondtype	Face value million DKK	Number of issues	Face value (%)	No. of issues (%)
Non-callable bonds	393,979	359	31.76	16.95
Callable mortgage-backed bonds	651,578	1,713	52.53	80.88
Index-linked bonds	121,421	11	9.79	0.52
Adjustable rate bonds*	73,502	35	5.93	1.65
Total	1,240,480	2,118	100.00	100.00

1 As shown in chapter 5 the prepayment option is equivalent to a call option held by the mortgage-owner.

2 Source: Numbers are drawn from the official database of the CSE, May 7 1992. Foreign currency bonds as well as lottery bonds have been excluded. *) Includes 14 MBB issues totalling 2,095 million DKK.

The pricing model is based on a so-called arbitrage-free term structure model in which the current zero coupon yield curve combined with an estimate of future interest rate volatility determine the possible stochastic evolution of future interest rates. In this approach bonds with interest dependent cash flows are priced relative to current market prices of non-callable fixed-payment bonds. The arbitrage-free model is extended to an after-tax setting.

Chapter 2 discusses the estimation of zero coupon term structures on the Danish bond market using weekly data for the period January 1985 to July 1992. It is shown how the appropriate estimation techniques have changed in the period due to the lack of long term non-callable bonds. Starting in 1988 the results point to an increase in the efficiency of the government bond market compared to the 1985-87 period.

In chapter 3 we review the arbitrage-free models of the term structure and a general no-arbitrage MBB pricing model is presented which depends on the characteristics of the MBB, the stochastic evolution of the future interest rates and a so-called prepayment function. The prepayment function determines the fraction of mortgage holders who prepay their loan in a single period. By varying the specification of this function, different pricing models arise.

A substantial part of the Danish literature on MBBs has analysed how the tax system influences the prepayment behaviour of the individual borrowers. Chapter 4 contains a discussion of bond market equilibrium with differential taxation. As shown by several authors unlimited tax-arbitrage possibilities may arise in a bond market, if institutional investors, fully taxed of capital gains as well as interest income, can trade freely with private investors, who are taxed of interest income only. An institutional environment is proposed, in which no unlimited tax-arbitrage possibilities exists even though short-sale is permitted for bonds of all maturities. The institutional restrictions needed can be seen as a strict version of the current Danish minimum interest rate rules. The chapter extends the analysis to a stochastic setting and develops a computationally efficient modification of the arbitrage-free models, which allows for the calculation of pre-tax as well as after-tax prices of options. The resulting model is used for all after-tax calculations in the following.

The remaining chapters are concerned with the specification and estimation of the prepayment function. Chapter 5 studies optimal prepayment behaviour. The prepayment option is viewed as an American call-option on a non-callable mortgage. Using the theory of American options a critical level of interest rates is determined at which a rational mortgage holder would exercise the option. The chapter shows how optimal prepayment behaviour depends on coupon rate, time to maturity, prepayment costs, the tax status of the individual borrower as well as the special Danish cash-loan system.

In a separate analysis the prepayment model is extended to allow for the so-called *delivery option* i.e. the possibility of a change to a higher coupon loan for tax reasons. It is shown that the value of the delivery option is due to the existence of differentially taxed investors. A dynamic debt-management model is proposed in which the rational borrower determines the optimal exercise of the delivery as well as the prepayment option. The delivery option is shown to have a minor effect on prices for MBBs through its influence on prepayment behaviour.

The chapter finally shows that a simple American option model in which all borrowers prepay simultaneously implies discontinuities in the estimated prices. This makes the model unsuitable as the basis of a MBB pricing model.

Any empirical useful model for MBBs must incorporate the heterogeneity of individual mortgage holders and the fact that their prepayment behaviour is determined by many factors not all of which can be explained by rational behaviour. In chapter 6 we develop a model in which the individual mortgage holder prepays his mortgage when a stated required gain (RG) is reached. The required gain is assumed to be normally distributed across individual borrowers. Different kinds of prepayment behaviour can be described by different choices of the prepayment distribution.

The payments from the MBB can be found as follows: At any given point in time the mortgage holders observe the gain implied by the current level of the term structure. Borrowers, for whom the observed gain exceeds the required gain, prepay their loans, while the rest continue their scheduled payments. The prepayment rate can thus be found from the specified mean and standard deviation of the required gain distribution. Contrary to the American option model of chapter 5 in which the prepayment rate was either zero or 100%, the RG-model allows for a continuum of prepayment rates.

The chapter states the different prepayment motives and discusses how they can be represented in the required gain model. The dependency of estimated prices on the level of interest rates, the volatility, the current shape of the initial term structure, the time to maturity, the required gain distribution and the tax rate is analysed and the results are compared to the American option model of chapter 4. It is shown that optimal prepayment behaviour can be incorporated if the RG-distribution is made dependent on remaining time to maturity and perhaps other parameters as well.

Chapter 6 continues with a discussion of duration and convexity measures for MBBs. It is shown that negative convexity makes fairly priced MBBs inferior to non-callable bonds of the same duration except for small changes in interest rates. The chapter ends with an empirical example in which the difference in prepayment rates from three similar MBB-issues is attributed to partly unobserved differences in the composition of borrowers. As a by-product the example shows some potential with respect to empirical use.

Chapter 7 contains the development, estimation and test of a complete pricing model for Danish mortgage-backed bonds. We use a version of the required gain model of chapter 6 in which the prepayment behaviour is estimated from observed prepayment rates. The estimation is based on a newly constructed data-set consisting of published quarterly prepayment rates for the period 1988-92. It is the first empirical prepayment model for the Danish market, but similar models have in recent years been developed for the US market.

Chapter 7 starts by a survey of pricing models proposed for the US mortgage backed securities market. The prepayment functions of the US-models are estimated on actual prepayment experience explained by the history of interest rates as well as individual characteristics of the mortgage pool in question. Contractual differences exists between the US and the Danish market and a set of explanatory variables is proposed, which allows for the special features of the Danish mortgage system.

The survey is followed by a brief summary of the data-set. Contrary to most US-models the RG-model is based on a 'micro-economic' decision model for the individual mortgage holder. It is shown how the decision model can be restated as a so-called probit model and how the behaviorial parameters can be inferred from observed prepayment data. Various specifications are tested and it is concluded that a simple four parameter model provides a fair description of the historical prepayment behaviour from Danish MBBs. More sophisticated specifications should be tried as more observations becomes available. The empirical analysis identifies a group of high-risk bonds which probably contains a large share of corporate borrowers.

The estimated prepayment function represents a handy summary of the historical reaction of Danish mortgage holders to the changing level of interest rates. This knowledge of borrowers behaviour is used in the general arbitrage-free bond pricing model together with estimates of zero coupon yield curves and volatility. We derive some estimates of prices and durations for a sample of 33 MBBs issued by Nykredit. Prices and durations are calculated every four week in the period 1988 to 1992. The pricing results indicate that MBBs have been relatively cheap for most of the period although the general correspondence is close. We would thus expect a high return from these bonds relative to non-callable bonds with similar risk.

The hypothesis is tested against a new data-set of four-weekly holding period returns (HPR) for the period 1988 to 1992. HPR have been calculated for the 33 MBBs as well as for all large non-callable government bonds. Annual plots of average return against standard deviation show that the returns from the MBBs follow the prediction of the model quite closely. These findings are confirmed by a preliminary regression model on the full data set. The regression procedure allow us to adjust for the differences in interest rate risk as well as the difference in non-callable yield. The preliminar results indicates that average HPR for the 10%, 12% and especially 11% MBBs have been above HPR for bonds of similar risk and non-callable yield.

It is furthermore shown the duration measures derived from the MBB pricing model explains changes in HPR with a precision equivalent to the duration measures for non-callable bonds although a simple adjustment must be used to compare the two. This suggest that the MBB durations can be very useful in the calculation of hedge-ratios, portfolio risk measures etc.

The regression finally indicates that active trading based on net present value estimates from the pricing models could contribute considerably to increased performance from MBBs as well as non-callable bonds.

The conclusion of the thesis is that the model developed provides a good description of the Danish market for mortgage backed bonds. The model is based on arbitrage-free pricing principles and it is tested on available data for prepayment rates, market prices and holding period returns. The empirical approach provides a common foundation for the discussion, test and use of these pricing models. As the estimation techniques evolve and more data becomes available we expect the MBBs to be priced on a routine basis with a degree of precision close to non-callable bonds.

2 The Danish Term Structure 1985-1992

In this chapter we will discuss the estimation of zero coupon yield curves on Danish bond data and present results based on weekly data in the period January 1985 to July 1992. Yield curve estimates is at the center of any kind of empirical bond analysis and our estimates will later be used as input in the pricing model for mortgage backed bonds developed in the following chapters.

Bond pricing models represents an important application but term structure estimates may also contain information on the formation of expectations regarding future market conditions. For an analysis of the time series dynamics of the Danish term structure we refer to Tanggaard(1992), Tanggaard and Engsted (1992), who test different versions of the expectations hypothesis using a cointegration approach.

A previous paper, Jakobsen and Tanggaard (1988), discussed yield curve estimation results for the period from January 1985 to August 1988. Since 1988 several changes have occurred in the Danish bond market. The number of official bond price quotations has fallen due to the introduction of computerized trading systems. Secondly mortgage prepayments driven by the low level of interest rates have affected the prices of the callable mortgage backed bonds (MBB) used to estimate long term yields. Both changes makes it hard to obtain valid estimates on long term yields. On the positive side an increase in the efficiency of the market for government bonds has improved estimates of short term zero coupon yields .

The yield curve estimation procedure is discussed in section 2.1. Section 2.2 addresses the problems of sample selection and we end up by suggesting three different models. The empirical estimates are compared in section 2.3. Section 2.4 shows some evidence on the increased pricing efficiency of the Danish market for government bonds. Section 2.5 contains the conclusions and some suggestions for further research.

2.1 The statistical model

The basic statistical framework for the estimation of zero coupon yield curves is now broadly accepted and for a full development of the economic arbitrage arguments behind this model we refer to the literature[3].

This section gives a brief résumé of the model used in Tanggaard and Jakobsen (1988a) (TJ-88). It is a cross section type of model that prices bonds relative to each other on one particular trading day[4]. Assume a sample of N default-free, fixed-payment[5] bonds with an $N \times 1$ vector of market prices P. Let $m = 1...M$ be an index of all settlement dates t_m in the sample and let B be the $N \times M$ non-stochastic payoff matrix with entries of zero if the bond has no payment at date m. D is defined as the $M \times 1$ vector of discount rates where a single element $d_m \equiv d(t_m)$ denotes the present value of one DKK delivered at date t_m. Finally we define the $N \times 1$ vector of *present values* PV as the product of the payoff matrix B and the vector of discount rates D, i.e. $PV = B \cdot D$.

In a perfect market the no-arbitrage condition implies an equality between P and PV. The interpretation is that payments due at the same date t_m must be priced at the same price $d(t_m)$ across all bonds. In a perfect and complete market any deviation from this 'Law of One Price' leads to risk free arbitrage opportunities. In real world markets however various pricing errors must be taken into account. We assume that prices are equal to their present values plus some additive error ε, i.e.

(2.1) $$P = B \cdot D + \varepsilon$$

The pricing errors ε_n, $n = 1,..,N$, are assumed to be independently normal distributed with heterogeneous variance σ_n^2. Chambers et.al.(1984) propose and Tanggaard and Jakobsen(1987) further investigate the specification

(2.2) $$\sigma_n^2 = \gamma^2 \cdot T_n^\delta$$

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3 The first part of this section is taken from Jakobsen and Tanggaard(1988). McCulloch(1971) and Carleton and Cooper(1976) were the first to apply regression techniques to the estimation of zero coupon yields from available data on coupon bearing bonds. Our present model has been inspired by the model of Chambers et.al.(1984). Tanggaard and Jakobsen(1988) and Tanggaard(1992) contain a full description of the cubic spline methodology with a fuller set of references to the international literature.

4 To simplify notation we have suppressed the dependence of all prices, cash flows and estimates on the trading day used.

5 The terms 'fixed-payment' or 'non-callable' bond refer to bonds in which the future cash flow is known with certainty as opposed for instance to the callable mortgage backed bonds analysed in later chapters.

where T_n is the time to maturity of bond n, γ^2 denotes that part of total variance, which is unrelated to time to maturity and δ determines the degree of which T_n affects variance.

Setting $\delta = 0$ gives the special case of homogeneous variance, but as shown in Tanggaard and Jakobsen (1987) this specification is strongly rejected on the Danish bond market, in favour of a specification in which the variance increases with time to maturity. A fixed value of $\delta = 1$ as proposed in TJ-87 has been used throughout the estimations[6] indicating that variance increases linearly with time to maturity.

To estimate the model a parameterisation of the individual discount rates $d(t_m)$ is needed. We have chosen the following specification

$$(2.3) \qquad d(t) = \exp(-R(t) \cdot t)$$

in which the continuously compounded yield curve $R(t)$ is taken to be a cubic spline[7]. To be more specific the time scale is divided into segments $[\tau_{h-1}, \tau_h]$. The points $\tau_h, \quad h = 1, \ldots, G$ are called knots. The cubic spline is defined as a twice continuously differentiable function which coincides with a polynomial of degree at most three in each of the G segments. This definition leaves a total of $G + 3$ independent parameters. To complete the definition one can choose two a priori restrictions, cf. below. Given these limitations it is possible to parameterize the cubic spline by its values in the G+1 knots.

Different versions of the cubic spline model result according to the choice of a priori restrictions. Two different versions referred to as the *flat spline* and the *soft spline* will be used in the present paper. To get the *soft spline* the second derivative at τ_0 is set to zero. As shown in Jakobsen and Tanggaard(1988) (JT-88) this reduces the short rate flexibility and thereby increases reliability of short rate estimates. The second restriction $R(\tau_{G-1}) = R(\tau_G)$ restricts the term structure estimates of two last knots to be equal. This reduces long rate flexibility but yield curve estimates between τ_{G-1} and τ_G may vary somewhat. The soft spline has a total of $G + 1$ independent parameters.

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6 Other measures like duration could have been used to adjust for heteroscedasticity, as done in JT-88.

7 Readers interested in a formal derivation of the cubic spline models are referred to Tanggaard and Jakobsen (1988a).

If the moments of τ_{G-1} and τ_G are set to zero one obtains the *linear spline* discussed in the above mentioned paper. In this model the cubic spline is linear in the last segment. Finally the *flat spline* emerges, if we restrict the linear spline, so that $R(\tau_{G-1}) = R(\tau_G)$. The flat spline has a total of G independent parameters.

As discussed in JT-88 the flat spline provides more reliable long rate estimates, but with an important caveat. If the flat segment starts ’too early’ one may distort the pricing of mid-term bonds. It is therefore very important that the sample contains bonds with a long enough maturity to allow the flat segment to start at say 18 years.

Despite its complicated definition the cubic spline parameterization proposed in TJ-88a is linear in the parameters $R(\tau_k)$ and in its unrestricted forms it can fit any shape of yield curve. Compared to e.g. standard polynomials the correlation between estimated parameters tends to be low, which allows for a relatively independent estimation of different yield curve segments.

A range of alternative parameterizations have been proposed in the literature. Tanggaard and Jakobsen(1988b) use a non-nested test methodology to compare some of the most popular ones including the discount function splines of McCulloch(1976), the yield curve polynomials of Chambers et.al.(1984), various exponential formulations as well as the one-factor model of Cox, Ingersoll and Ross(1985) (CIR). The analysis concluded that the cubic splines, even in a flat spline version, gave the best overall fit to bond prices.

The CIR model emerges as the solution to a one-factor equilibrium model and provided the equilibrium model is a fair description of reality it may have some a priori advantages. The assumptions of the equilibrium model include a constant short term volatility and a fixed long term interest rate. These assumptions are hard to justify in practice at least for nominal bonds. Several authors including Brown and Dybvig(1986) and Barone et.al.(1991) have used the CIR specification, but allowed the parameters to be estimated on daily cross section data using a statistical specification like (2.1). The absence of intertemporal time-series restrictions on the parameters conflicts with the assumptions of the CIR model and used in this way the CIR-specification is just a non-linear functional form on line with several others[8].

To estimate the cubic spline model we have used the maximum likelihood approach described in TJ-88a. Outliers are eliminated by a two-step procedure. First the estimation is performed on the full sample and standardized residuals are calculated according to the heteroscedastic specification (2.2) above. Bonds with residuals above 2 times the adjusted standard deviation are then eliminated, and a new estimation is performed on the reduced sample[9].

As shown in JT-88 short term estimates from the flat spline are very volatile, while the soft splines[10] obtains more stable short rate estimates. To improve the flat-spline model we apply a two-step procedure. First a soft spline is used to estimate the short term interest rate, and secondly the flat spline is estimated with $R(0)$ fixed at its soft spline value[11].

2.2 Problems of sample selection

To estimate the term structure of interest rates prices and cash flows for a sample of bonds are needed. The ideal sample should consist of high liquidity bonds, distributed throughout the maturity spectrum and void of any obstacles due to tax considerations or call features.

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8 Barone et.al.(1991) estimate the CIR-specification using a sample of Italian government bonds with a maximum maturity of 7 years. As expected their estimates on long term yields are highly volatile. One of the ’attractions’ of the CIR-model is that it possesses a limited set of shapes. Estimated yield curves are therefore always aesthetically pleasing but the drawback may be that the model misprices bonds of certain maturities. Some evidence on mispricing is given in the Barone et.al. and Brown and Dybvig papers. We intend to investigate the pricing ability of the CIR-model in a later paper using Danish bond market data.

9 Bonds with less than three month remaining are always excluded due to illiquidity and rounding errors, cf. JT-88.

10 The short-end restriction of the soft spline corresponds to the *normal spline* model analysed in JT-88.

11 The computations were done using RIO ver. 2.1, which is a PC-program for the estimation and application of zero coupon yield curves. Different versions of RIO have been used since 1986 by almost all major Danish financial institutions. The system has been jointly developed by Carsten Tanggaard and the author, cf. Jakobsen and Tanggaard(1992).

Figure 2.1 shows the total number of official bond price quotations on the Copenhagen Stock Exchange for each Wednesday in the period January 1985 to July 1992. There is a sharp reduction in the number of official price quotations around January 1989 due to the introduction of a decentralized electronic trading system. Before 1989 the official price was defined as the latest bid at the close of trade, while the official price in the electronic trade system is defined as a daily average of market prices[12]. This institutional change has made fewer bond quotations available for estimation purposes.

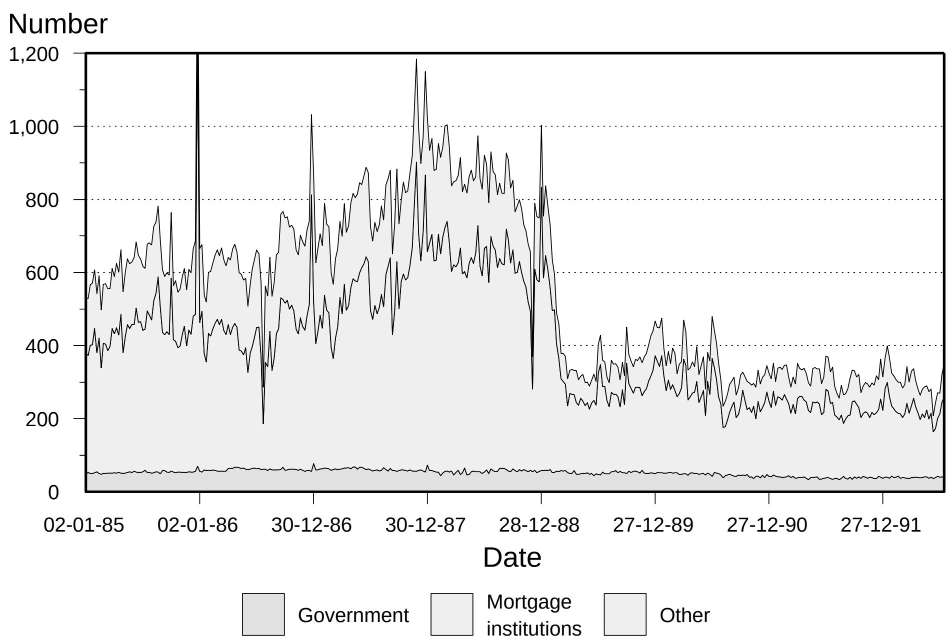

Figure 2.1: The number of official bond price quotations at the Copenhagen Stock Exchang e,1985 to 1992.

Table 2.1 gives an overview of bonds listed at the Copenhagen Stock Exchange as of May 7 1992. Numbers are given in millions DKK outstanding face value for different assumptions on callability, coupon rates and time to maturity. Corporate bonds, floating rate and indexed linked bonds, bonds denominated in foreign currencies as well as bond issues below DKK 100 million in outstanding face value have been excluded in the table and from the following sample selection. This leaves us with a total of 644 bonds.

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[12] At the previous open outcry auction dealers traded one bond at a time, which increased focus on less liquid issues. Note that a latest bid price could be quoted even if no trading took place. In the electronic trading system dealers have to watch all bonds simultanously, and obvious technical restrictions have forced dealers to concentrate on the more liquid issues.

Table 2.1: *Outstanding face value in million DKK of selected bonds listed at the Copenhagen Stock Exchange (CSE), May 7 1992, grouped by callability, years to maturity and coupon rate. Source: Calculations based on the CSE database.*

			Non-Callable bonds				
Maturity	0-5	5-10	10-15	15-25	25-	Total	%
Coupon							
%	23,795	2,133	14,950	1,657	348	42,883	4.24
<= 8	152,877	80,983	3,124	2,967	0	239,951	23.70
9	36,482	14,061	18,329	6,852	0	75,724	7.48
10	229	665	298	2,002	985	4,179	0.41
11	0	24,097	0	1,063	0	25,160	2.49
12							
Total	213,383	121,939	36,701	14,541	1,333	387,897	38.32
%	21.08	12.05	3.63	1.44	0.13	38.32	
No. of bonds	70	47	18	23	6	164	
			Callable mortgage backed bonds				
Maturity	0-5	5-10	10-15	15-25	25-	Total	%
Coupon							
<= 8	368	4,012	9,204	12,632	15,939	42,155	4.16
9	1,560	1,662	38,443	57,797	71,174	170,636	16.86
10	2,367	27,033	93,921	162,785	41,433	327,539	32.36
11	0	577	0	18,818	10,003	29,398	2.90
12	0	11,551	12,893	23,192	6,987	54,623	5.40
Total	4,295	44,835	154,461	275,224	145,536	624,351	61.68
%	0.42	4.43	15.26	27.19	14.38	61.68	61.68
No. of bonds	20	66	84	162	148	480	

Callable mortgage backed bonds accounts for 62% of all large bond issues. The remaining 38% as measured by face value is non-callable bonds, mainly issued by the Danish government. Long term bonds above 10 years are dominated by MBB issues which accounts for 92% of all outstanding face value. Non-callable bonds have an equally dominating 87% share of maturities below 10 years. Tabulations done on other dates throughout the sample period show a similar pattern.

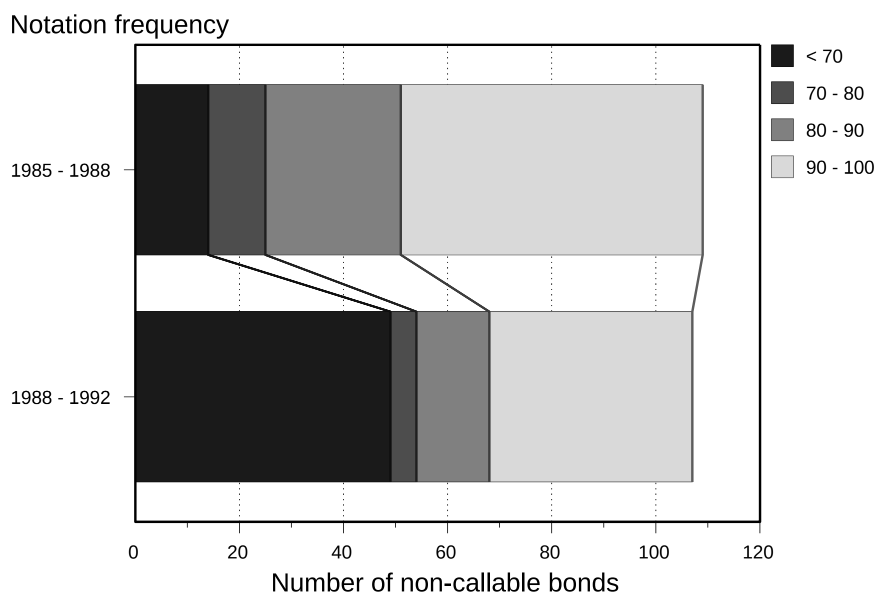


Figure 2.2: Plot of notation frequency for large non-callable bonds.

To estimate the term structure we have selected two different samples. The first sample consists of non-callable government bonds supplemented by long non-callable bonds from the mortgage credit institution DLR. Coupon rates below 9% are excluded to avoid biases introduced by taxation, cf. section 4.2. Finally we require a notation frequency above 75%[13]. A total of 59 bonds passed this test somewhere between January 85 to June 88 while 55 bonds participated in the remaining period.

As shown below the final estimations on the non-callable sample includes only 1-2 bonds above 13 years of maturity. The second sample was drawn according to the same criteria but a number of 9% MBBs with maturities above 13 year were included to allow the estimation of long term yields. Higher coupon MBBs were not included due to prepayment risk. The mixed sample contained 86 bonds in the period 85-88 and 82 bonds in the period 88-92.

[13] The calculation of notation frequency divides the number of official bond quotations by the number of days for which the bond has been traded. As shown in Sørensen(1991) this requirement reduces estimation errors considerably.

2.3 Overview of the empirical results

The following section reviews estimation results for three different models. On the non-callable sample a soft spline (NC-SS) as well as a flat spline model (NC-FS) has been estimated, both with knots fixed at 0, 2, 5, 13 and 30 years. For the mixed sample we have estimated a flat spline model (MS-FS) with knots at 0, 3, 8, 18 and 30 years.

Table 2.2: *Average estimation results from the three term structure models*

	No. of estim.	Std. dev	Gamma	Sample-size	Outliers	No. of bonds in segment			
Non-callable						0-2	2-5	5-13	13-30
Soft spline	393	0.34	0.163	27.1	1.6	10.7	8.0	7.0	1.5
Flat spline	393	0.46	0.192	27.1	1.6	10.7	8.0	7.0	1.5
Mixed sample						0-3	3-8	8-18	18-30
Flat spline	393	0.67	0.217	45.3	2.2	14.0	8.9	6.5	8.9

Average estimation results are shown in table 2.2. Standard errors are fairly small. Average errors are lowest for the non-callable soft spline and highest for the mixed sample flat spline. This is partly caused by differences in sample-maturity. Comparing the maturity-adjusted standard deviation gamma, it is seen that the MS-FS-model is roughly equivalent to the non-callable flat-spline model. The NC-SS gamma-value of 0.163 implies an average standard deviation of 0.163 (16 basis point) for one-year bond prices and a 52 basis point average pricing error on 10 year bonds.

Yield curve estimates from the three models are shown in figures 2.3-10 while figures 2.11 and 2.12 contain plots of standard deviation and gamma[14]. Term structure changes can be summarized by the non-callable flat spline estimates on 0, 5 and 20 year yields shown in figure 2.3. According to this plot a number of shifts have taken place[15]. Term

[14] All yield estimates use annual compounding. The plots of standard deviation and gamma has been smoothed by a 6 week simple rolling average.

[15] An explanation of term structure changes according to changes in macroeconomic variables would be highly relevant, but it is outside the scope of the current paper. Readers are referred to The Quarterly Bulletin from the National Bank of Denmark.

structure estimates starts January 1985 with an upward sloping yield curve. In the period from January 1985 to April 1986 a general fall in the level accompanied by a downward change in slope takes place ending April 1986 at an almost flat term structure around 9%. In summer 1986 the market returns to an upward sloping yield curve and the level increases sharply. 1987 is rather stable with small parallel shifts being the main attraction. During 1988 there is an almost parallel downward shift with short rates reaching a bottom around 8%. 1989 is very volatile with estimated short rates changing upwards more than 4 percentage point. In October 1988 the market changes overnight to a downward sloping yield curve. In February 1990 an increase in long rates leads to a flat yield curve, but from January 1991 up to June 1992 long rates fall while short rates increase resulting in a downward sloping yield curve at a very low level of long rates. The last few observations show some sign of an increase in long rates, possibly caused by the outcome of the EEC referendum.

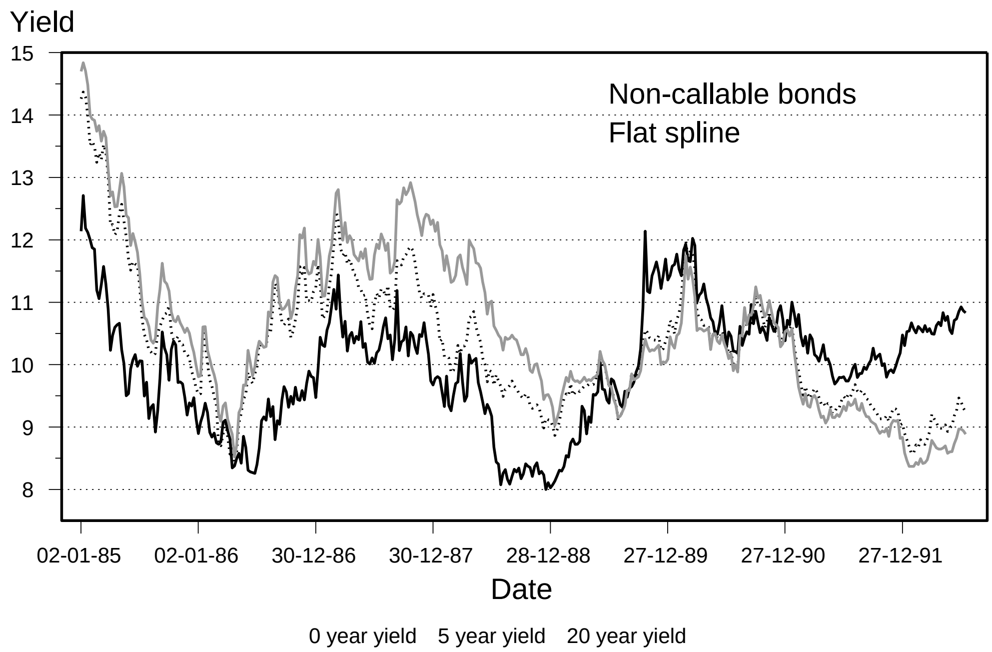
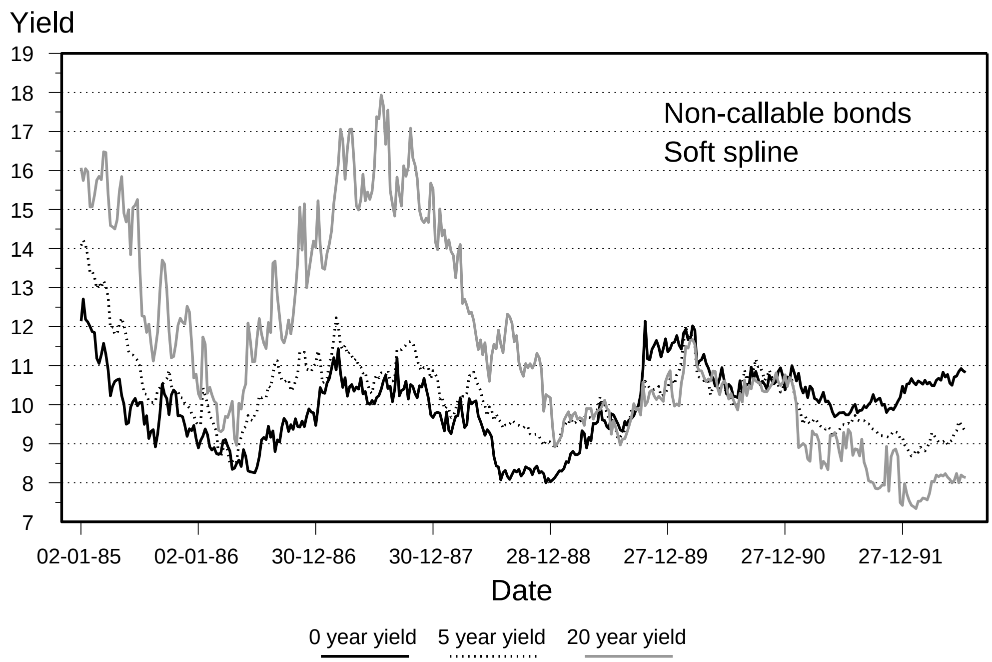


Figure 2.3: Term structure estimates for the non-callable flat spline model.



Figur 2. 4: Term structure estimates for the non -callable soft spline model.

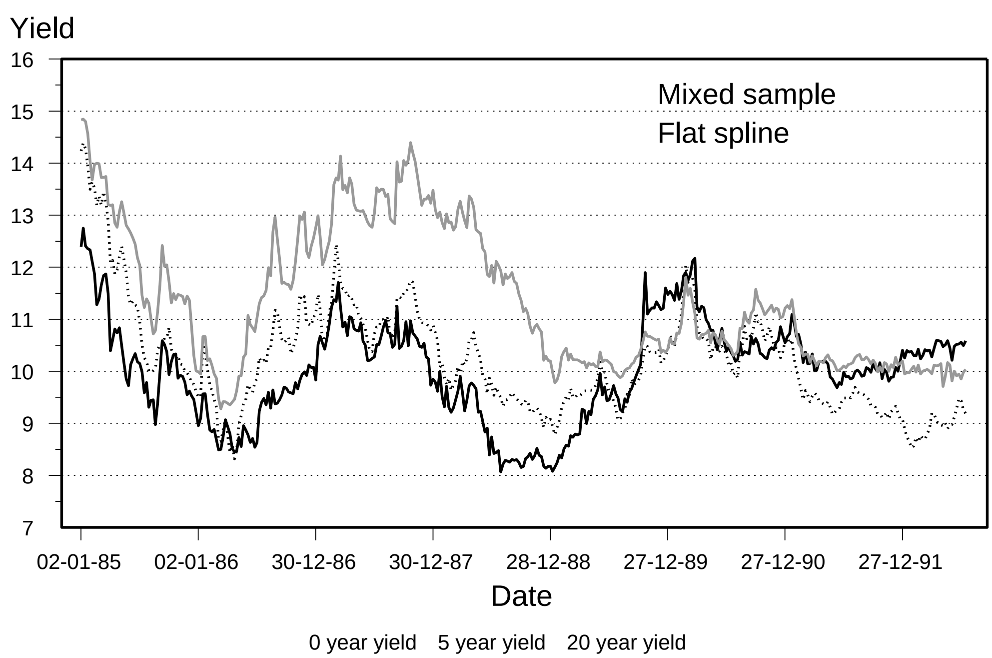


Figure 2.5: Term structure estimates for the mixed sample flat spline model.

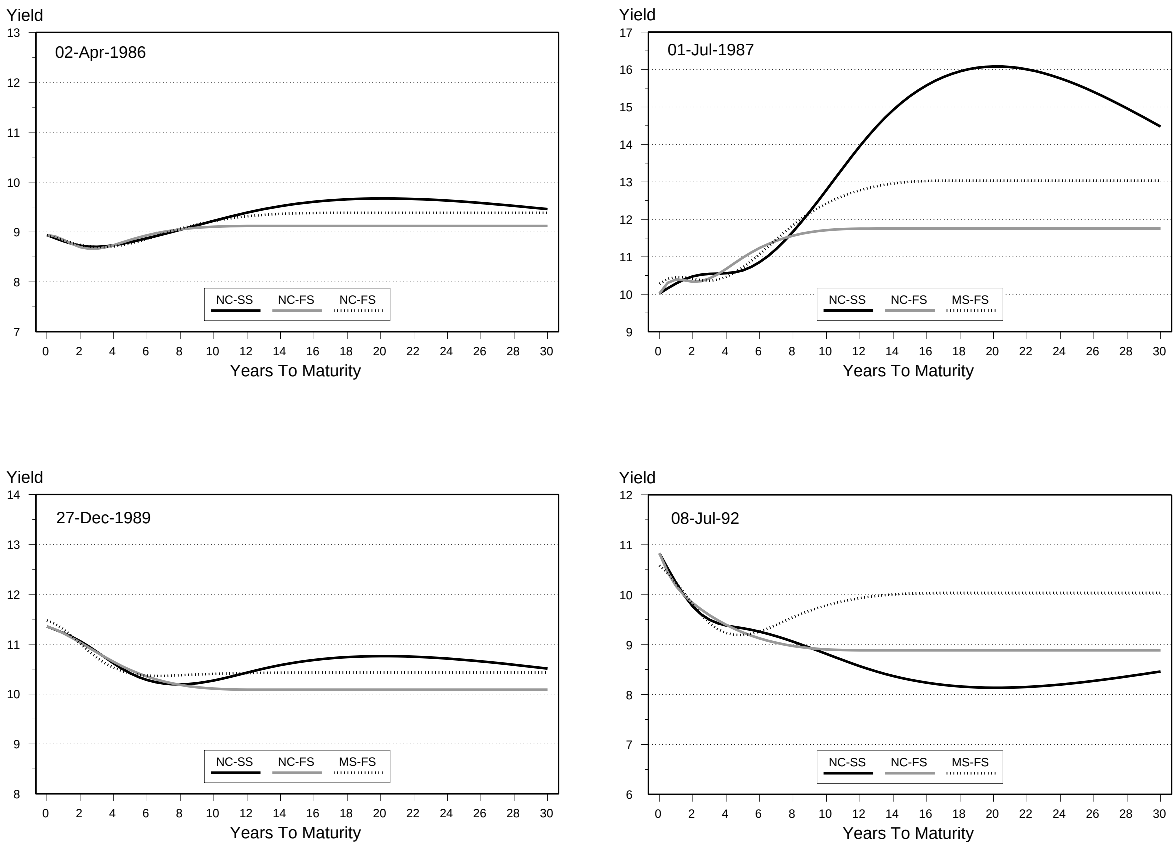


Figure 2.6: Comparison of yield curve estimates on different dates.

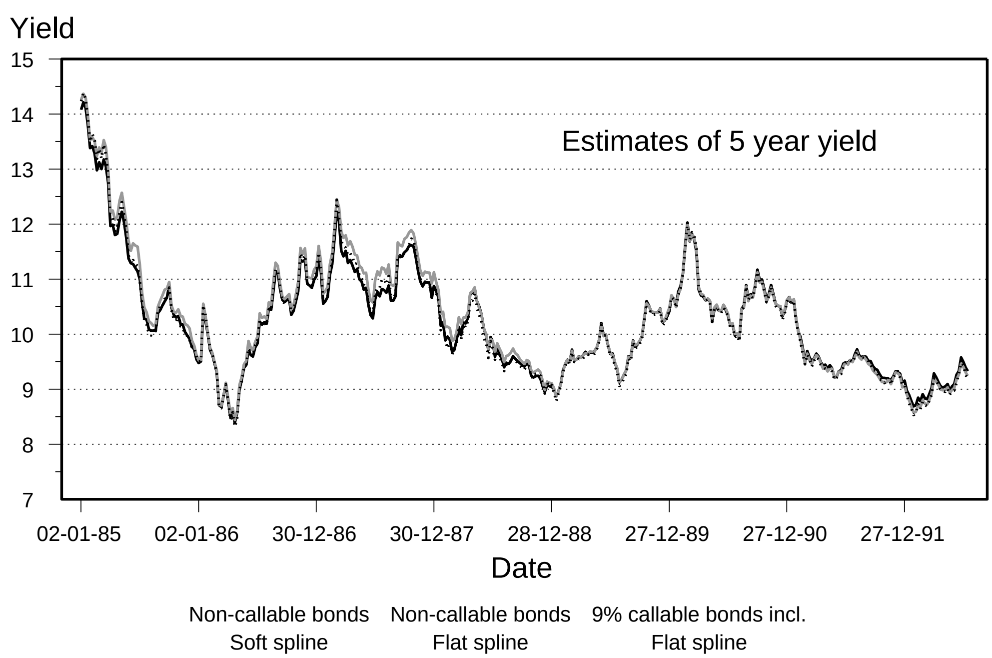


Figure 2.7: Comparison of 5 year yield.

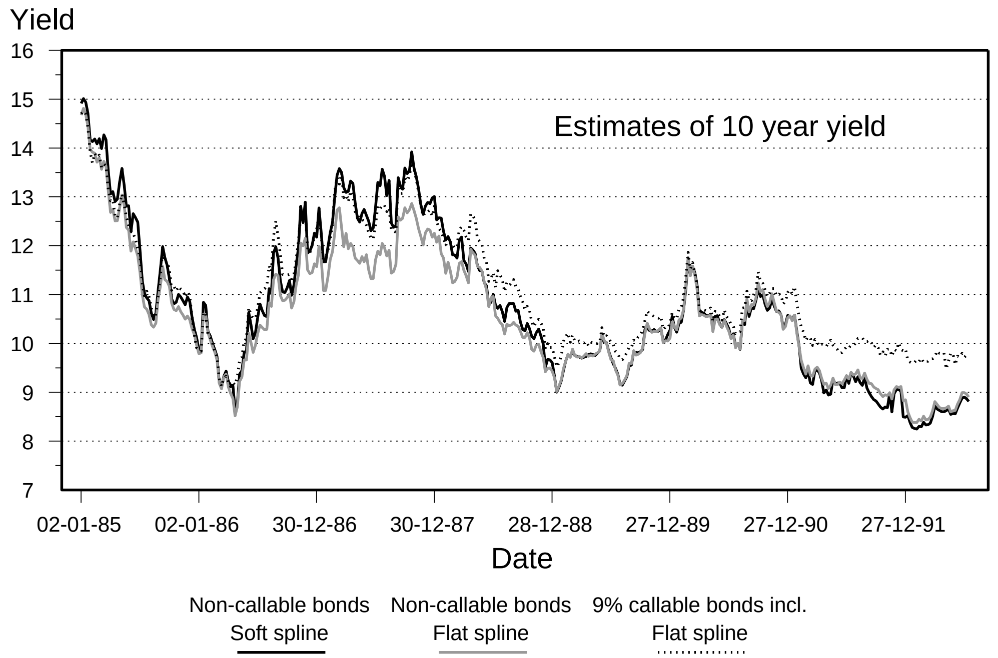


Figure 2.8: Comparison of 10 year yield.

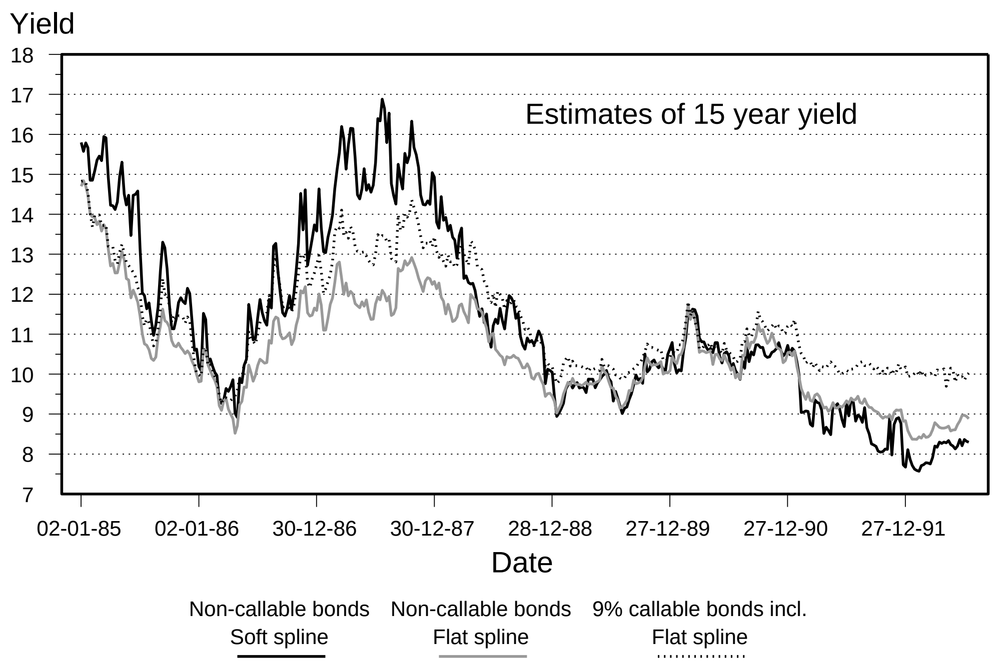


Figure 2.9: Comparison of 15 year yield.

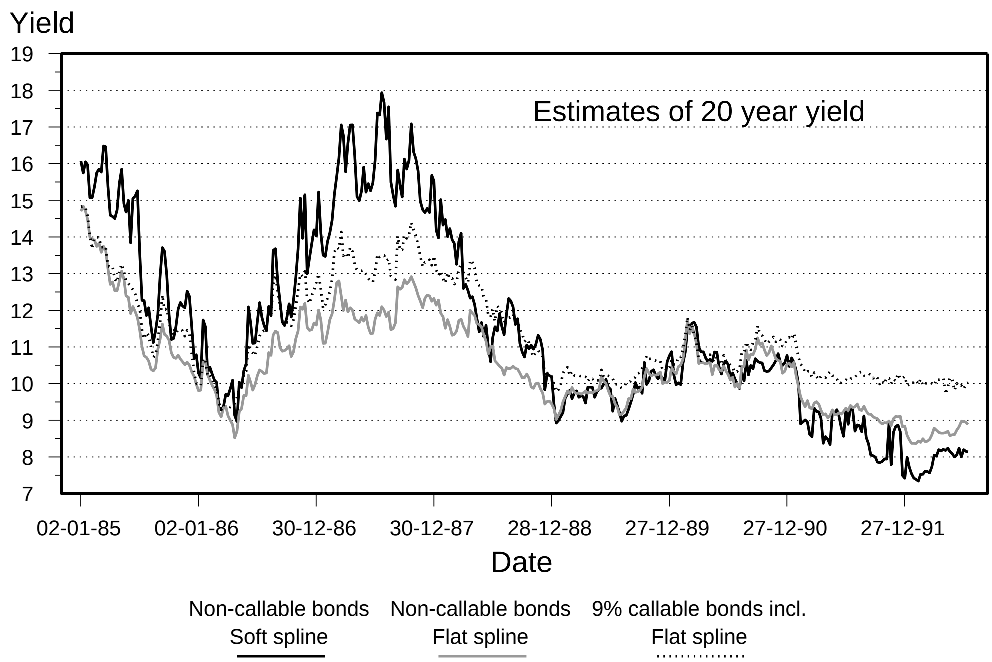


Figure 2.10: Comparison of 20 year yield.

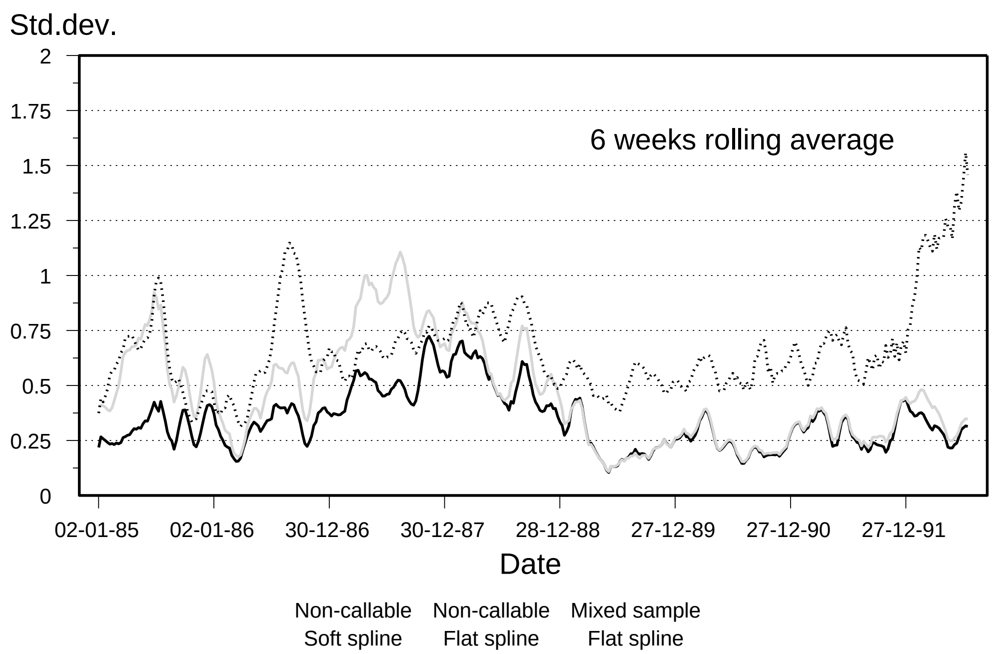


Figure 2.11: Plot of estimated standard deviation, smoothed by a 6 weeks moving average.

The non-callable soft-spline model (NC-SS) consistently shows the lowest pricing errors for the sample of non-callable bonds. That can be attributed to the very high flexibility of the soft spline. But as a predictor of longer term yields the model performs badly. Compared to the flat spline models the NC-SS model has very little a priori restriction on long term shape and the non-callable sample contains no

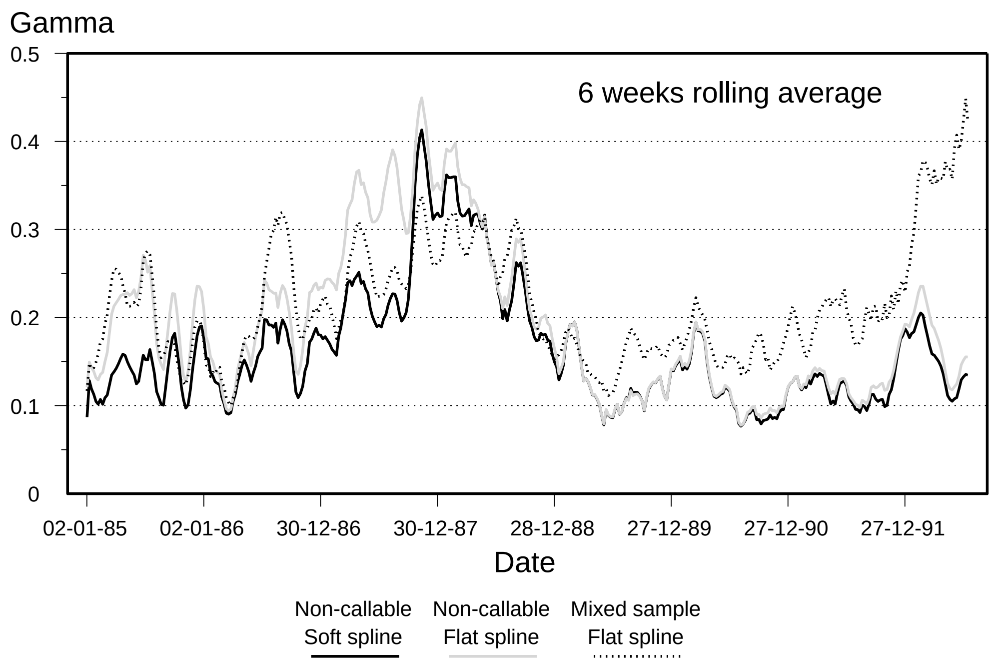


Figure 2.12: Plot of estimated gamma, smoothed by a 6 weeks rolling average.

information on long term yields beyond 13 years. This explains the erratic long term yield estimates shown in figure 2.4 We can conclude that the NC-SS should not be used if long term estimates are needed.

The non-callable flat-spline model (NC-FS) was restricted to a constant yield from 13 years of maturity. This leads to stable estimates on long term yields, but as mentioned in section 2.1 the flat segment may start to early. For the period 1989 to 1992 the model seems to do very well with standard errors almost identical to the NC-SS model. For the period from 85-88 the model performs worse, especially in 1987. In this period yield curves are mostly upward sloping and the constant yield restriction results in large standard errors and gamma values, even compared to the mixed sample model. We can conclude that the constant yield restriction starting at 13 years is well suited for the 89-92 regime, but the NC-FS model should not be used for the pre-88 data.

The mixed-sample flat-spline model (MS-FS) includes long term MBBs which allow the estimation of long term rates. Compared to the NC-FS model the constant yield restriction starts at 18 years, and we would expect this model to be more applicable in periods of upward sloping yield curves. This expectation is confirmed by the plots. In the period 87-88 gamma as well as 5 and 10 year yield estimates for the MS-FS model compares closely with the NC-SS model and long term estimates from the MS-FS is far more stable. In periods with high prepayment risks we would expect an upward bias in long term estimates as the 9% MBBs should be priced below similar non-callable bonds. Comparing with NC-FS estimates for the period 1991-92 we see a spread between long term estimates of approximately 100 basis point. This spread can be attributed to prepayment risks. A similar bias is not found in the short period of low interest rates around April 1986. It seems as if the low rate period has been too short for prepayment risk to enter 9% MBB-prices.

To summarize we cannot select one of the three models as the uniformly best model for all purposes or for the entire sample period. The NC-SS provides the closest fit to the sample of non-callable bonds, but its long term estimates are poor. For analysis of yields up to 10 year the NC-SS model is very well suited however. For longer term yields the flat-spline models are needed. In periods with low prepayment risks the MS-FS is probably the best model but its long term estimates becomes upward biased at low levels of interests. In these periods the NC-FS could be used instead.

Several routes could be taken to improve the model. One could search for a parameterization which provides stable long term estimates with no distortion of mid-term bond prices. Alternatively one could hope for more long non-callable bonds to enter the market and recent steps taken by the mortgage credit institutions seems to point in this direction. And finally one could try to calculate the value of the prepayment option and adjust MBB-prices accordingly.

2.4 The efficiency of the Danish bond market

The plots of standard deviation and gamma for the non-callable sample show a sharp reduction in pricing error beginning in the middle of 1988. This could indicate an increase in market efficiency.

Figure 2.13 compares weekly estimates of net present value (NPV) defined as the difference between present value and market price for two different government bonds. Present value estimates are obtained from the NC-SS model. The bond labelled ’12% Ser 2001’ is a 12% serial bond maturing in 2001, while ’10% Stl. 1994’ is a 10% ordinary bond maturing 1994. The two bonds have comparative Macaulay durations for most of the period.

The 12-2001 serial bond seems to be consistently undervalued for most of the period up to mid 1988. The ordinary 10-1994 bond seems equally overvalued with market prices up to 400 basis points above present value. From mid 1988 the mispricing disappears and NPV for the two bonds lies close to zero for the rest of the period.

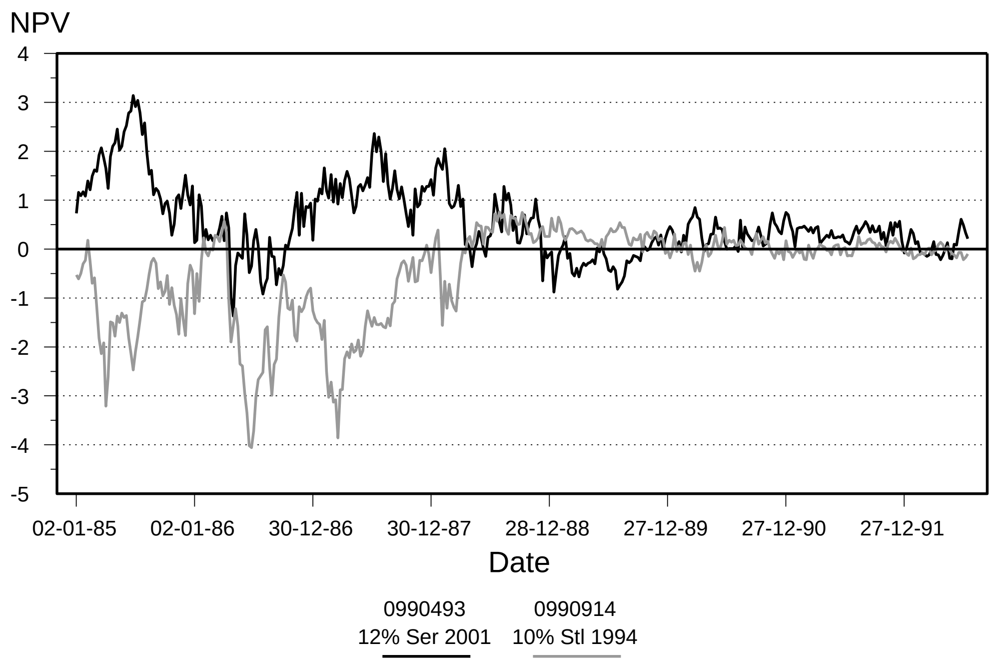


Figure 2.13: Net present value for two different government bonds

The undervaluation of serial loans relative to ordinary loans is of more general nature as discussed in the papers by Jakobsen and Tanggaard(1987,1989). Our main explanation has been that traders tend to use a simple yield-to-maturity comparison between bonds of equal maturities as their prime instrument for the assessment of relative value. In periods of upward sloping yield curves this would explain the observed mispricing. When yield curves flatten the mispricing disappears. Other factors like the broad acceptance of zero coupon yield curve techniques and the introduction of an electronic trading systems may have helped to increase efficiency further.

2.5 Conclusion

This chapter has dealt with the estimation of zero coupon yield curves on the Danish bond market for the period 1985-1992. The primary conclusion has been the following.

Estimation of yield curves up to 10 years can be done with rather high precision using a sample of non-callable government bonds.

Estimates of longer term yields are harder to come by due to the lack of large long term non-callable bonds. We have suggested the use of a mixed sample in which the non-callable bonds have been supplemented by 9% callable mortgage backed bonds. This model does well in the period from 85-90, but in the remaining part of the sample period the 9% MBBs are subject to a downward price bias caused by prepayment risks. For this part of the sample period a flat spline model estimated on non-callable bonds seems a better choice.

Estimation errors showed an increase in market efficiency starting 1988. Part of this increase could be explained by the flattening of the yield curve, but the widespread acceptance of more sophisticated trading techniques may also be an important factor.

The estimated yield curves can be used as the basis for more sophisticated pricing models. This is the subject of the following chapters.

3 Arbitrage-Free Pricing Models

The last chapter was devoted to the estimation of yield curves for fixed-payment non-callable bonds. These bonds take a 32% share of the Danish bond market with DKK 394 billion of outstanding face value[16]. The remaining part of the bond market consist of bonds in which the cash-flow depends on the future development of interest rates. This includes the 53% in callable mortgage backed bonds as well as the adjustable rate bonds taking up 6%. To price such issues a stochastic term structure model is needed. The same applies to contracts traded in closely connected markets like options on bonds, swaps, caps, floors, collars etc.

The purpose of the present chapter is to develop an arbitrage-free pricing model for Danish callable mortgage backed bonds (MBBs). The model could use any kind of stochastic term structure model, but we have chosen the model by Black, Derman and Toy(1990) (BDT), which is a simple arbitrage-free, one-factor model using a multiplicative binomial assumption on the short term interest rate distribution. The BDT-model is probably one of the most widely used discrete time stochastic term structure models.

Section 3.1 contains a review of arbitrage-free pricing models with special emphasis on the BDT-model. In section 3.2 it is shown how well-known interest rate dependent securities like options on bonds and callable bonds can be priced by an arbitrage-free model. Finally section 3.3 contains a description of MBBs and develops a framework for the pricing of these rather complicated securities. The detailed specification of this model is the subject of later chapters.

3.1 Arbitrage-free models of the term structure

Stochastic term structure models describes the future stochastic evolution of the term structure as driven by one or more factors. One-factor models typically use the short term interest rate as the underlying source of interest rate risk, while two-factor models may include short rate volatility, long rates or yield spreads as a second source of risk.

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[16] The figures are taken from the database distributed by the Copenhagen Stock Exchange as of May 7, 1992. Indexed linked bonds as well as bonds issued in foreign currencies have been excluded.

As all asset risk can be attributed to the same basic factors, the well-known no-arbitrage argument restricts the price behaviour in a way which can be used to price any asset relative to the underlying factors.

The traditional approach[17] specifies a stochastic model of the underlying factors and deduces the possible term structures from these data. The traditional models are typically estimated on historical time series data. A problem with this approach is that the models may not be able to fit the current term structure.

The arbitrage-free models[18], initiated by the Ho and Lee(1986) paper, take available market data, like the term structure as input and restrict the stochastic process of the underlying factors in such a way that the no-arbitrage condition is consistent with these data. Accordingly the derived stochastic process may be unable to fit historical time series data.

Both types of models have their strengths. If one is concerned with the *explanation* of the current term structure the traditional approach would be appropriate, but if the main purpose is to price derivative assets relative to prices of non-callable bonds the arbitrage-free models are more convenient.

While the arbitrage-free models by Ho and Lee(1986) (HL) and Black, Derman and Toy(1990) (BDT) used a binomial setting, it is now evident that most popular continuous time stochastic models could be expressed as an arbitrage-free model as well, cf. the papers by Hull and White(1990a,1990c) and Jamshidian(1991).

The current section contains a short review of the BDT-model. The section is meant for reference mainly and no new results will be developed. For an introduction we refer to the original BDT(1990) paper. Section 3.1.1 presents the general no-arbitrage condition in a binomial setting. The two most important interest rate models is

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[17] One-factor models representing the traditional approach can be found in Vasicek(1977), Dothan(1978) and Cox, Ingersoll and Ross(1985), while Brennan and Schwartz(1979) as well as Longstaff and Schwartz(1991) are examples of two-factor models.

[18] Other examples of arbitrage-free models can be found in Jamshidian(1987), Black, Derman and Toy (1990), Pedersen, Shiu and Thorlacius (1989), Heath, Jarrow and Morton (1990), Hull and White(1990a). Hull and White(1990c) introduce a very general trinomial setup, which incorporates most of the models proposed so far. Jakobsen and Jørgensen(1991) focus on the common structure of the Ho and Lee and the BDT-model.

reviewed in section 3.1.2, while 3.1.3 contains a discussion of the estimation of volatilities. Section 3.1.4 goes through the numerical calibration procedure. The forward induction method used in this thesis is rather new and a full exposition is given.

3.1.1 The no-arbitrage condition

The binomial term structure model can be defined as follows. Assume the time scale (measured in years) to be divided into intervals of length dt starting at $t = 0$. At each period n (starting at time $t = n \cdot dt$) there is a total of $n+1$ different states of the world, numbered $s = 0, 1, .., n$[19]. The combination of date n and state s is called the *date-event* (n, s). In the period following date-event (n, s) either an 'up-state' occurs leading to date-event $(n+1, s+1)$ or a 'down-state' leading to $(n+1, s)$. $r(n, s)$ denotes the short rate, defined as the annualized continuously compounded risk free rate of interest at (n, s). The price at date-event (n, s) of a zero-coupon bond maturing at date n+1 is denoted by $p(n, s) \equiv \exp(-r(n, s)dt)$.

Restating the well-known no-arbitrage option valuation argument of Cox, Ross and Rubinstein(1979) it can be shown that the price, $V_i(n, s)$, of any risky asset i at date-event (n, s) must apply to the following general backward equation

$$V_i(n,s) = p(n,s)\{\theta(n,s)V_i(n+1,s+1) + (1-\theta(n,s))V_i(n+1,s)\} \quad . \tag{3.1}$$

The backward equation states that the price at date-event (n,s) should be obtained as a weighted average of up- and down-state prices, discounted by the risk free rate of interest. To avoid arbitrage possibilities the same weight $\theta(n, s)$ must be used across all securities. $p(n, s)\theta(n, s)$ could be interpreted as the market price of up-state money at (n, s), while $p(n, s)(1-\theta(n, s))$ is the corresponding price of down-state money. A security paying 1 unit in both states will have a price equal to the sum of up- and down-state price that is $p(n, s)$ and an annualized return equal to $r(n, s)$. In general these state-prices will depend on the overall risk preferences of the market. Note that (3.1) is similar to a risk-neutral valuation equation except that the weight θ has no direct relation to the objective probability of up-state. In the special case of an

19 To avoid unnecessary notation the discussion is limited to the binomial model. The notation here is adapted from Jamshidian(1991). Readers interested in a more general setup, which connects more readily to continuous time models, are referred to the paper by Jamshidian or Hull and White-(1990c).

economy of risk-neutral investors only, the parameter θ must equal the probabilities of up- and down-state. The parameter θ is often referred to as the risk-neutral or martingale probabilities.

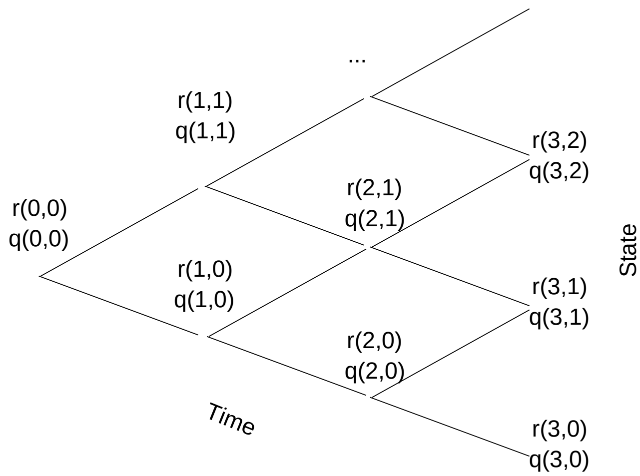

Figure 3.1: The structure of the binomial lattice ($q \equiv \theta$)

If the values of $\theta(n,s)$ and $r(n,s)$ were known at any future date-event, then (3.1) could be used to price a wide range of assets. For some examples see BDT(1990). As a special case one could price zero-coupon bonds of any maturity, and thereby derive the entire term-structure.

The backward equation (3.1) applies to the traditional as well as the arbitrage-free models. The difference lies in the determination of θ relative to the stochastic process for the risk-free interest rate. The traditional term structure models start with a known drift and volatility for the risk free interest rate and use an exogenous assumption on risk preferences to determine θ[20]. Nothing ensures that modelled prices of zero-coupon bonds matches the current term structure.

[20] Hull and White(1990a) develop the relationship between the traditional and the arbitrage-free models in a continuous time setting. The papers by Jensen and Nielsen(1991) and Jakobsen and Jørgensen(1991,pp.19ff) discuss the same issue in the binomial model.

The arbitrage-free models always fit the current yield curve. These models start with a known volatility and an assumed value for θ and then a drift of the future risk free interest rate is constructed so that modelled zero-coupon bond prices match the observed term structure. In the following we use the arbitrage-free approach[21].

3.1.2 Examples of arbitrage-free models

In its most general form the backward equation (3.1) introduces two parameters, $r(n,s)$ and $\theta(n,s)$ for each date-event (n,s). As a typical pricing lattice for callable mortgage backed bonds may contain say 120 periods leading to $121 \cdot 122/2 = 7.381$ different date-events, some additional structure is needed. In this section we outline two of the most popular parameterizations.

The generalized Ho and Lee (1986) model (HL) assumes[22] a constant value of $\theta(n,s)$ and an additive relationship between short rates of different states that is

$$\text{(3.2)}\qquad \begin{aligned} &a) \quad \theta(n,s) \equiv \theta \\ &b) \quad r(n,s+1) = r(n,s) + h_n \quad \Leftrightarrow \quad r(n,s) = r(n,0) + s \cdot h_n \end{aligned}$$

The parameters needed are now reduced to the constant θ and two values $r(n,0)$ and h_n for each period n.

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21 A full examination of the validity of the arbitrage-free approach could (easily) fill several papers, but a simple analogy might help.

Arbitrage-free models, like the BDT-model, price assets dependent on the future short-rate relative to prices of zero coupon bonds in the same way as the Cox-Ross-Rubinstein model prices options dependent on future stock-prices relative to the current market price of the stock. In both models an observed volatility, σ, is transformed into a multiplicative spread by the relationship $\delta = 2 \cdot \exp(-\sigma \cdot \sqrt{\Delta t})$. Using this spread a lattice of the underlying risky asset is constructed in which the 'risk-neutral' pricing formula (3.1) fits observed market price(s). The constructed lattice can then be used to price any derivative asset. In none of these models will the true drift of the underlying asset enter the calculations. Only risk-adjusted up- and down-state prices are needed and these can be inferred from available market prices.

22 The original article of Ho and Lee(1986) is stated in terms of arbitrage-free shifts in the initial discount function, but as proved in Jakobsen and Jørgensen(1991) their approach is equivalent to (3.2). Ho and Lee(1986) used a time independent additive spread. The use of a time dependent spread h_t is a slight generalisation first proposed by Pedersen, Shiu and Thorlacius(1989).

The Black, Derman and Toy(1990) model employs a multiplicative spread between short rates. Their lattice structure can be stated as

$$\text{(3.3)}\qquad \begin{aligned} &a) \quad \theta(n,s) \equiv \theta \\ &b) \quad r(n,s+1) = \delta_n \cdot r(n,s) \quad \Leftrightarrow \quad r(n,s) = \delta_n^s \cdot r(n,0) \end{aligned}$$

The BDT-model contains the same number of parameters as the HL-model, but some differences remain. As shown by Jamshidian(1988) the HL-model converges to a continuous time model with interest rates being normally distributed. There will thus exist a positive probability of negative interest rates. This applies to the binomial version as well. The BDT-model converges toward a log-normal distribution for the short-term interest rate and the probability of negative interest rates is zero. Secondly the spot-rate variance of the HL-model is state-independent, while the spot-rate variance of the BDT-model is proportional to the level of the spot-rate. These differences has lead many researchers and practitioners to prefer the BDT-model. This is the approach taken in the current thesis as well.

Works by several authors, especially Jamshidian(1987,1988,1990) and Hull and White(1990a) may change the current trend back toward the HL-model. The HL-model and its continuous time version is analytically far more tractable than the BDT-model and closed-formed pricing formulas have been given for a wide range of derivative assets, even including American options on coupon bearing bonds. The problem of negative interest rates could be kept at an insignificant level by the introduction of a positive rate of mean reversion. As interest rates of all maturities are normally distributed the statistical estimation of the model will be well-defined. On the contrary no closed-form pricing formulas exists for the BDT-model and the statistical distribution of longer term interest rates is not a simple well-known distribution[23]. These features may redefine the mean-reverting HL-model as the standard arbitrage-free term-structure model, cf. Hull and White(1992).

3.1.3 Specification of interest rate volatility

As shown in Jamshidian(1991) the BDT-model can be viewed as a discretization of the following continuous time lognormal model.

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[23] For a simple proof see the appendix of Jakobsen and Jørgensen(1991).

(3.4) $$d(\log r) = [b(t) + \log(r)\sigma'(t)/\sigma(t)]dt + \sigma(t)dz$$

where $z(t)$ is a Wiener process, with normal distributed increments dz, $\sigma(t)$ denotes a deterministic time dependent volatility of the future short rate and b(t) is a deterministic time dependent drift term determined from the initial term structure.

To freely use the BDT-model for different time-steps *dt*, it is necessary to establish the connection between the volatility $\sigma(t)$ and the spread parameter δ_n. From date-event (n,s) the log-rate i.e. the logarithm of the spot rate $r(n,s)$, may change to either $\log r(n+1,s)$ with martingale "probability" θ or to $\log r(n+1,s+1)$ with probability $1-\theta$. The variance, $V(t,s)$, of the log-rate in this binomial process is given by

(3.5) $$VAR(n,s) = \theta(1-\theta)(\log r(n+1,s+1) - \log r(n+1,s))^2 = \theta(1-\theta)\log\delta_n^2 \quad .$$

In the continuous time lognormal model the variance of the log-rate for a small period dt is given by $\sigma(t)^2 \cdot dt$. Setting (3.4) equal to (3.5)[24] and rearranging gives the following formula for δ_n

(3.6) $$\delta_n = \exp\left(\frac{\sigma(n \cdot dt)\sqrt{dt}}{\sqrt{\theta(1-\theta)}}\right) \quad .$$

The continuous time limit of the generalized Ho-Lee model is given by (cf. Jamshidian(1991))

(3.7) $$dr = [b(t) + r\sigma'(t)/\sigma(t)]dt + \sigma(t)dz$$

where the drift term $b(t)$ and the volatility of absolute yield changes $\sigma(t)$ are time dependent functions.

By a similar argument the spread-parameter h_n is found by

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[24] The volatility estimate will typically be based on historical time series using actual probabilities, while the binomial variance estimate uses risk-neutral probabilities θ. In the limit this distinction disappears cf. Madan, Milne and Shefrin(1988) for a discussion.

(3.8) $$h_n = \frac{\sigma(n \cdot dt)\sqrt{dt}}{\sqrt{\theta(1-\theta)}}$$

cf. Jamshidian.(1990,p.57).

3.1.4 The numerical calibration procedure

The ability to match a specified initial term structure is the distinguishing feature of arbitrage-free models. The match is accomplished through an adjustment of the expected drift of future short term rates. For some models, like the Ho and Lee, explicit formulas can be found which relates expected future short term rates to the specified initial term structure, but in general a numerical calibration procedure must be used. This section contains a detailed and notationally simplified exposition of the forward induction method introduced by Jamshidian(1991)[25]. The forward induction approach is compared to the ’naive’ method of backward induction.

In the last section the spread parameters, δ_n, were assigned to an exogenous volatility estimate and θ and $r(n,0)$ remain as unknown parameters. Unless otherwise stated we set $\theta = 0.5$ in the following[26].

Let $P(t)$ denote the initial term structure of zero coupon bonds. $P(t)$ is typically found by a statistical estimation procedure similar to the one described in chapter 2. For use in the binomial model $P_n \equiv P(n \cdot dt)$ is defined as the price at time zero of a zero coupon bond maturing at period *n*. The purpose of the calibration procedure is to determine the bottom-rates $r(n,0)$ so that modelled prices of a *n*-period zero coupon bond equals the observed price P_n[27].

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[25] The numerical procedure given here applies to all arbitrage-free models including the HL-model. A version of the forward induction method will later be used to calibrate an after-tax lattice to the after-tax term structure, cf. section 4.3 below.

[26] θ could be assigned other values between 0 and 1 as well, but in the limit pricing results remain the same. Setting $\theta = 0.5$ ensures the fastest rate of convergence, cf. Jamshidian (1990, p.57).

[27] Note that the bottom-rate $r(n,0)$ determines all short-term rates at date n through the spread parameter δ_n. An increase in $r(n,0)$ will increase $r(n,s)$ for all states s and thus increase the expected n-period rate. The calibration could of course be parameterized by say the median rate instead. Numerical stability depends on the total range of short term rates and the exact choice does not seem to matter.

The calibration is done by either backward or forward induction. The backward induction method is the least involved of the two. Step 1 ($n = 0$) obtains the first bottom-rate $r(0,0)$ from P_1 by observing that $P_1 = p(0,0) = \exp(-r(0,0)dt)$, which implies that $r(0,0) = -\ln(P_1)/dt$[28].

The general step for $n > 0$ chooses $r(n,0)$ to match P_{n+1}. Here an iterative procedure is used[29]. Assume $r(k,0)$ to be determined for $k = 0, 1, ..n-1$. One starts with a guess of $r(n,0)$. This guess determines $p(n,s)$ for all s=0,..,n. The cash flow at time n+1 of the $n+1$-period zero-coupon bond can now be discounted back to time zero using the backward equation (3.1) at each date-event. By comparing the resulting price to the observed price P_{n+1} one obtains an improved guess. The process is iterated until a value of $r(n,0)$ is reached, at which the calculated price equals the observed price[30]. Then the whole procedure moves to period n+1.

The backward induction method quickly calls for a faster computer. For each of the values $r(n,0)$ one needs several passes back through the entire lattice. The number of nodes is proportional to n^2. The total computation time needed for a N-period lattice is thus proportional to N^3.

The forward induction method introduced by Jamshidian(1991) is an ingenious solution procedure, which speeds up calculations dramatically. The method uses the theoretical concept of an Arrow-Debreu security. Let $G(n,s)$, denote the price at time zero of the (n,s)-Arrow-Debreu security, defined as a security with a cash flow of unity at date-event (n,s) and zero elsewhere. In the binomial model any security with a time- and state-dependent cash flow could be viewed as a portfolio of AD-securities in the same way as fixed-payment bonds are portfolios of zero-coupon bonds.

The concept of AD-prices was touched above when noted that $\theta \cdot p(0,0)$ equals the market price of up-state money and $(1-\theta) \cdot p(0,0)$ the price of down-state money.

28 As mentioned above $r(n,s)$ denotes annualized continuously compounded interest rates. Our computer implementation uses non-annualized rates $\tilde{r}(n,s)$ with simple compounding, i.e. $p(n,s) = 1/(1+\tilde{r}(n,s))$, to increase efficiency.

29 As an exception $r(1,0)$ could be solved for analytically.

30 The derivative with respect to $r(n,0)$ can be calculated, which enables the use of a Newton-Raphson procedure.

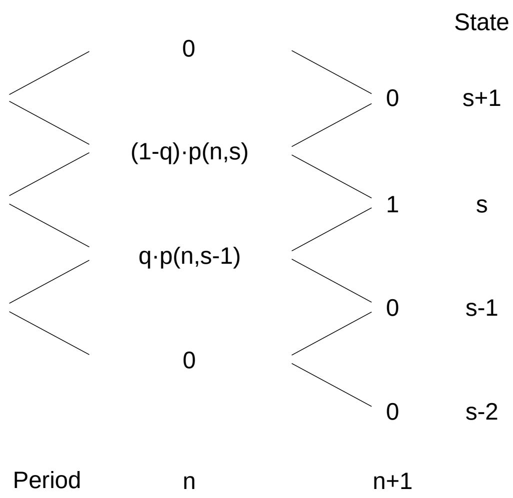

Figure 3.2: The value of an (n+1,s)-AD-security ($q \equiv \theta$).

Extending this argument figure (3.2) shows the value in period n of an $(n+1,s)$-AD security. In period n the security is worth $\theta \cdot p(n,s-1)$ in state s-1, $(1-\theta)p(n,s)$ in state s and the value is zero elsewhere.

As a simple consequence one gets the so-called forward equation:

$$G(n+1,s) = (1-\theta)p(n,s)G(n,s) + \theta \cdot p(n,s-1)G(n,s-1) \tag{3.9}$$

The interpretation of (3.9) should be straightforward from figure 3.2. The value at time zero of an (n+1,s)-AD-security is equal to its value in period n times the respective AD-prices in period n. The equation works for the boundary states as well, if we define $G(n,-1) \equiv G(n,n+1) \equiv 0$ and $p(n,-1) \equiv p(n,n+1) \equiv 0$.

The first step ($n = 0$) of the Jamshidian forward induction method calculates $r(0,0)$ to match P_1. This is done directly as shown above. The AD-prices at time 1 is calculated as $G(1,0) = (1-\theta)p(0,0)$ and $G(1,1) = \theta \cdot p(0,0)$. The general step determines $r(n,0)$ for $n > 0$ to match P_{n+1}. On outset all n-period AD-prices, $G(n,s)$ are assumed to be known from the previous step. To connect AD-prices with zero-coupon prices note that the price at time zero of an $n+1$-period zero coupon bond can be expressed as a sum of AD-prices

$$P_{n+1} = \sum_{s=0}^{n+1} G(n+1,s) \tag{3.10}$$

Using the forward equation (3.9) P_{n+1} is expressed by the sum of known values $G(n,s)$ multiplied by $p(n,s)$:

(3.11)
$$P_{n+1} = \sum_{s=0}^{n} p(n,s)G(n,s)$$

Relation (3.11) is an equation with one unknown variable, $r(n,0)$. Each choice of $r(n,0)$ determines the vector of one-period discount factors $p(n,s)$ and the r.h.s. value can be compared to the observed zero-coupon price P_{n+1}. The equation is easily solved in 2-4 iterations by a Newton-Raphson procedure.

To summarize the forward induction method starts at time 0 by determining $r(0,0)$ and then progresses forward in the lattice. In step *n* the bottom-rate $r(n,0)$ is solved for numerically by use of (3.11) and $G(n+1,s)$ is then determined for all *s* by the forward equation (3.9). The advantage of the forward induction method is that each iteration of equation (3.11) involves only values at time *n* as compared to the backward induction method in which a single iteration needs a recalculation of the entire lattice. Computation time used to solve for *N* different time periods is therefore proportional to N^2 as compared to N^3 for the backward induction method. One could say that *n*-period AD-prices is a sufficient statistic for the entire lattice structure between time 0 and *n*.

For good reasons the forward induction method has been used throughout this thesis. One of our earlier implementations of the BDT-model used backward induction. In its most optimized version with explicit derivatives, a Newton-Raphson search-procedure, simple compounding, ’smart’ initial guesses etc., a 120 period lattice could be calculated in a few minutes[31]. The corresponding 360 period lattice took hours. A simple change to the Jamshidian(1991) forward induction method reduced computation time to 4 seconds for a 120 period lattice, while a 360 period lattice is done in 20 seconds. For the simulations done in this thesis forward induction has probably saved several months of effective computer time.

[31] Calculations was done on a 20 MHz 386 PC with a mathematical coprocessor. A state-of-the-art PC/workstation might improve computation times by a factor of 10, but even that would become slow.

3.2 Pricing techniques

Explicit pricing formulas have been developed for some combinations of assets and interest rate processes. In the absense of explicit solutions one must resort to an approximative valuation using the binomial arbitrage-free lattice constructed in the previous section. The current section shows how the lattice can be used to value fixed-payment bonds, options on bonds and bonds with embedded options. The section ends with a short discussion of path-dependency.

A *fixed-payment bond* can be described as a vector of settlement dates, $\{t_i\}, \quad i = 1, .., M$ measured in years[32] with a corresponding vector of cash flows $\{b_i\}$. $B(n, s)$ denotes the value of the bond at date-event (n, s). The value at maturity, $n_M = t_M/dt$, is found as $B(n_M, s) = b_M$ for all states $s = 0, 1, .., n_M$. At any date n before maturity, the general recursive relationship

(3.12) $$B(n, s) = F(n) + \frac{1}{2}p(n, s)\{B(n+1, s+1) + B(n+1, s)\} \quad \text{for } s = 0, 1, ..n$$

can be used, where $F(n)$ equals the payment, b_i, if n corresponds to a settlement date t_i, i.e. $n = t_i/dt$ for some i. If n is not a settlement date $F(n) = 0$. Applying (3.12) from date $n_M - 1$ to zero returns the bond value at time zero.

The use of a lattice to price a fixed-payment bond at time zero is of course computational inefficient, since the value could be found from the initial term structure by

(3.13) $$B = \sum_{i=1}^{M} b_i P(t_i) \qquad .$$

The arbitrage-free approach ensures that the two methods give the same result.

Consider now the value, $C(n, s)$, of a European option on a fixed-payment bond. The option matures at time T corresponding to trading date $n_T = T/dt$. To price the option one starts at the bonds maturity n_M and uses the backwards equation (3.12) to get the

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[32] In practice the payments may be due between the discrete trading dates of the binomial lattice, and some interpolation must be used, cf. section 3.3.

value of the bond, $B(n_T, s)$, at n_T for all states s. The contract features of the option translate the bond value into option values, $C(n_T, s)$, and finally the backward equation is used to discount these terminal option values back to date-event (0,0)[33].

The case of an American option on a fixed-payment bond is handled with little extra work. Let $C^*(n,s)$, which is a function of $B(n,s)$, denote the *exercise value* of the option[34]. The *hold-on* value, $C^+(n,s)$, is defined as the value of the option, if it is not exercised at date-event (*n,s*). By the standard value-maximizing argument, the value of the option is found by

(3.14) $$C(n,s) = \max\{C^*(n,s), C^+(n,s)\}$$

The pricing procedure first discounts the fixed-payment bond from n_M to $n_T + 1$. Option calculations start at n_T with $C(n_T + 1, s) = 0$. Each step n, $0 \le n \le n_T$, calculates $B(n,s)$ by (3.12). The exercise value $C^*(n,s)$ is found from the bond value and the hold-on value is given by $C^+(n,s) \equiv \frac{1}{2} p(n,s)\{C(n+1, s+1) + C(n+1, s)\}$. Finally relation (3.14) is used to get the option value $C(n,s)$.

Bonds with embedded American or European options, like callable corporate bonds, could be priced directly or as a portfolio of a fixed payment bond and an option. Let the cash flow, $F(n)$, from the underlying non-callable bond be defined as above. The direct approach denotes the value of the callable bond by $V(n,s)$, the redemption value, i.e. the value if the option is exercised, as $V^*(n,s)$ and the hold-on value by $V^+(n,s)$. Pricing starts at bond-maturity n_M setting $V(n_M, s) = b_M$. At any date-event n, $0 \le n < n_M$, we have $V^+(n,s) = F(n) + \frac{1}{2} p(n,s)\{V(n+1, s+1) + V(n+1, s)\}$. If n is a decision date for the bond issuer, $V^*(n,s)$ equals the contracted strike price plus the cash flow $F(n)$. The value of the callable bond can be found by

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[33] This is the procedure required by the BDT-model. For models of the Ho and Lee type with normally distributed interest rates $B(n_T, s)$ could be calculated by explicit formulas. This would increase efficiency by allowing the lattice to stop at n_T instead of n_M. Explicit formulas for European options of fixed-payment bonds exist as shown by Jamshidian(1989).

[34] For call options $C^*(n,s) = B(n,s) - X(n,s)$, while $C^*(n,s) = X(n,s) - B(n,s)$ for a put, where $X(n,s)$ denotes the possible time- and state-dependent strike price. In practice the calculation of C^* must take care of details like accrued interest, remaining face value of the underlying bond etc.

$V(n,s) = \min\{V^*(n,s), V^+(n,s)\}$. If n is not a decision date $V(n,s)$ equals $V^+(n,s)$. A putable bond is handled similarly except that $V(n,s) = \max\{V^*(n,s), V^+(n,s)\}$, if the put option can be exercised at time *n*.

The portfolio approach prices the option separately as discussed above. At time zero the embedded bond value is found as $V = B - C$ in case of a callable bond while V equals $B + C$ for a putable bond.

The cash flows of all assets discussed above did not depend on the history of interest rates leading to date-event (*n*,*s*). This allowed the use of a very efficient backward pricing algorithm. For some assets the cash-flow depends on the entire history of interest rates. For these *path-dependent securities* a different pricing procedure is needed.

Consider for example an adjustable rate bond, in which the coupon rate is fixed for 3 months at a size determined as an average of earlier short term rates. The value of this bond cannot be determined from the current term structure because the future coupon rates depends on future short term rates. The backwards pricing procedure does not work either, because the size of the coupon rate is unknown when discounting needs to be done. Derivative securities like caps give rise to a similar problem.

Callable mortgage backed bonds (MBBs) is a more complicated example cf. the discussion in the next section. The cash flow from a MBB depends on borrowers prepayment behaviour and in general this behaviour is partly determined by the history of interest rates.

From the theoretical point of view path-dependent pricing is easy. One picks a path, S, of short term rates, calculates the corresponding cash flow and discount it by the short-term rates. This price, V_S, constitutes the value of the security conditioned on interest rates evolving along S. The price at time zero is the obtained as a simple average across all available paths, i.e. $V = \sum_S V_S / N_S$.

As an unpleasant property of path-dependency the number of paths, N_S, equals 2^N, where N is the number of periods in the lattice. With lattices of more than say 15 periods the computations become prohibitive. One must therefore resort to Monte-Carlo techniques (MCT). Instead of averaging across all paths one averages across a randomly selected sample of paths. MCT therefore introduces a modelling error of its own, because the calculated price varies with the chosen sample. Selecting a larger sample decreases sample variance, but only very slowly[35].

Monte-Carlo techniques may be the only solution available for the valuation of certain types of securities, but algorithms tend to be complicated drawing heavily on intermediate storage and computer time. MCT's are furthermore restricted to European style options. The MBB-pricing models of this thesis are based on American-style option arguments and Monte-Carlo techniques will not be used. This is in contrast to most US-models, cf. the discussion of chapter 7.

To summarize we have shown how certain types of securities could be valued in a one-factor binomial lattice model. We have focused on some of the most important examples, but the arbitrage-free models could be extended to almost any kind of interest-rate dependent security. Of special interest for the current thesis is the callable mortgage backed bond, which will be analysed in the next section.

3.3 A MBB pricing model

This section introduces a pricing model for Danish mortgage backed bonds (MBBs). The price depends on the so-called prepayment function, which specifies the rate of prepayments as a function of time, the term structure and the characteristics of the mortgage pool in question. By changing the specification of the prepayment function several interesting models can be studied. This will be the subject of later chapters.

3.3.1 Description of the MBBs

As shown in section 2.2 more than 60% of the Danish bond market consists of fixed-coupon, callable mortgage backed bonds. MBBs are issued by mortgage credit institutions and each bond is backed by thousands of individual mortgages. Due to strict governmental regulation, joint and several liability and the use of real estate as collateral the default risk is insignificant. MBBs are so-called pass-through securities in that all payments from the individual mortgages are passed directly to the bond-

[35] To avoid choosing too large a sample variance reduction techniques can be used cf. Hull(1989,sec.9.1). Cheyette(1992) contains a very interesting discussion of Monte-Carlo techniques in the context of mortgage-backed bonds.

holders less a small administrative fee. Individual mortgages are highly standardized with coupon rate, settlement dates, amortization schedule and time to maturity being common to all mortgages in a single pool[36].

Most current MBBs are issued as either annuity or serial loans with quarterly payments. Before 1986 MBBs were issued as annuities with semi-annual payments. Nearly all mortgages are issued with either 20 or 30 years to maturity having coupon rates between 8 and 12%.

The complexity of MBB pricing stems from the prepayment option, which allow borrowers to prepay their loan at any time during the life of the mortgage. Prepayments are mainly driven by economic incentives, with prepayment rates being high when market rates are low. But compared to the callable bond analysed above one cannot rely entirely on rational behaviour. Prepayments can be driven by other factors like myopic liquidity concerns and real estate turn-over rates. Furthermore borrowers are heterogeneous differing in loan sizes, prepayment costs, tax-rates, available information and computational ability. These factors are captured in the *prepayment function*. In this section we present the prepayment function as a black-box model. The following chapters analyses various specifications.

An investor, who buys a mortgage backed bond, has in fact bought a share in a pool of individual mortgages. At each future settlement date the cash flow from his bond is equal to the total value of payments from the borrowers multiplied by his share of the remaining principal[37]. Ignoring minor differences in the pay-out ratios to each investor due to the amortization being done by lottery the future cash flow will be proportional to the initial principal.

The following notation is used to describe the callable mortgage backed bond. Let $k = 1, \dots, M$ be an index into the fixed set of settlement dates $\{t_k\}$. We assume that each date is measured as the number of years from time $t = 0$. $f = \{f_k\}$ is the vector of

[36] Time to maturity may vary somewhat in a single pool as the loans are issued over a period of 1-3 years.

[37] When the amortization is done by lottery there will of course be some uncertainty as to the exact share allocated to each investor. Because the lot size is small (DKK 1000) compared to the holdings of most professional investors this uncertainty will however be negligible.

repayments on the principal assuming no prepayments. The vector $c = \{c_k\}$ gives the corresponding interest payments. The vector of principal remaining at time t_k is denoted $g = \{g_k\}$, with $g_k = \sum_{j > k} f_j$. Repayments are normed so that $\sum f_k = g_0 = 1$

Consider a short period of time from t to $t + h$. In each such period an individual mortgage holder has the opportunity to either prepay or continue his loan. If a mortgagor decides to pay off his loan prior to maturity the remaining face value will be passed through to the investors at the earliest possible term date[38]. Let $\nu_k \leq t_k$ be the latest point in time where a decision to prepay the loan will result in the prepayment of face value g_k at date k. We shall refer to ν_k as the decision dates. The difference between t_k and ν_k is called *the term of notice* for termdate k. A decision to prepay at time t will become effective at the termdate $\kappa(t)$ defined by

(3.15) $$\kappa(t) \equiv \min\{k \mid (t \leq \nu_k) \vee (k = M)\}$$

At each termdate k in the interval from the prepayment decision up to and including termdate $\kappa(t)$ the debtor will continue to pay the expected cash flow of $c_k + f_k$.

Let $W(t)$ denote the value of the MBB assuming that all remaining debtors decide to prepay their loan at time t. In this case the cash flow from the MBB would be non-stochastic and $W(t)$ can be calculated directly from the term structure at time t, i.e.

(3.16) $$W(t) = \sum_{k \leq \kappa(t)} P(t, t_k) \cdot (f_k + c_k) + P(t, t_{\kappa(t)}) \cdot g_{\kappa(t)}$$

with $P(t, T)$ defined as the price at time t of a $T - t$ year zero coupon bond. Likewise we define $V^+(t, h)$ as the value at time t of the mortgage if no prepayment occurs in the period from t to $t + h$. $V^+(t, h)$ depends on the future stochastic prepayment behaviour of the remaining debtors.

Finally define the prepayment function $\lambda(t, h)$ as the fraction of the debtors, measured in terms of face value, who prepay their mortgages in the period from time t to time $t+h$. The remaining fraction $(1 - \lambda(t, h))$ of the debtors continues to the next period.

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[38] The lag between the prepayment decision and the time at which the investors receive the cash flow varies from 2 to 11 months.

The value of the mortgage at time t can thus be calculated as a portfolio of prepaying and non-prepaying mortgages:

$$V(t) = \lambda(t,h)W(t) + (1-\lambda(t,h))V^{+}(t) \tag{3.17}$$

To close the model we need a specification of the prepayment function as well as a stochastic pricing model.

3.3.2 The stochastic pricing model

Different term structure models have been employed for the valuation of mortgage backed bonds. Christensen (1985) uses the continuous one factor CIR-model in an extensive analysis of Danish MBBs, while the two-factor model by Brennan and Schwartz (1979) has been used by Schwartz and Tourus(1989) as well as McConnell and Singh (1991) in their analysis of US mortgage-backed securities. In this thesis we will use a binomial arbitrage-free model, as discussed in section 3.1. The solution procedure for a path-independent specification of λ is given below.

Rewriting the MBB-pricing model into a binomial lattice framework is straightforward. To avoid unnecessary notation we put $\theta(n,s) = \theta = 0.5$. The settlement dates t_k will in general occur between the discrete trading points n used in the binomial pricing model. To account for this, payments are discounted back to the nearest trading point with discounting done at the state-dependent risk free rate of interest. As an example consider the following mapping, which converts the vector of repayments on principal into a cash-flow $f(n,s)$ at each date-event (n,s):

$$f(n,s) = \begin{cases} f_k \cdot \exp(-r(n,s)(t_k/dt - n)) & \text{if } dt \cdot n \le t_k \le dt \cdot (n+1) \\ 0 & \text{otherwise} \end{cases} \tag{3.18}$$

Discounting could of course be omitted provided the time step dt is sufficiently small[39]. The interest payment $c(n,s)$ at date-event (n,s) is defined similarly to (3.18).

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[39] This way of interpolating the value of intermediate cash flows introduce state dependency into the otherwise state independent cash flow from the bond. Depending on implementation this increases computation times.

In the binomial model we define $\lambda(n, s, \alpha)$ to be the fraction of the mortgages prepaying on date-event (n, s) dependent on a vector of parameters α. Using this notation the value, $V(n, s)$, of the MBB can be written as:

$$V(n,s,\alpha) = \lambda(n,s,\alpha)W(n,s) + (1 - \lambda(n,s,\alpha))V^+(n,s,\alpha) \tag{3.19}$$

where $V^+(n, s, \alpha)$ is given by the formula

$$V^+(n,s,\alpha) = f(n,s) + c(n,s) + \frac{1}{2}p(n,s)\{V(n+1,s+1,\alpha) + V(n+1,s,\alpha)\} \tag{3.20}$$

Denoting the last date of the bond as $n_M = \text{int}\, t_M/dt$ the boundary condition becomes

$$V^+(n_M, s, \alpha) \equiv 0 \quad \text{for } s = 0, \ldots, n_M \tag{3.21}$$

The value in case of prepayment, $W(n, s)$ can be found as

$$W(n,s) = f(n,s) + c(n,s) + \frac{1}{2}p(n,s)\{W(n+1,s+1) + W(n+1,s)\} \tag{3.22}$$

At the boundary we set $W(n, s)$ equal to interest plus remaining face value

$$W(\kappa(n),s) = g(\kappa(n),s) + c(\kappa(n),s) \tag{3.23}$$

$\kappa(n)$ denotes the settlement date, at which the prepayment at time n becomes effective. Note that the boundary date $\kappa(n)$ changes, as we move backward in the lattice, and that calculations of $W(n, s)$ may overlap due to the term of notice.

If $\lambda(n, s, \alpha)$ is path-independent we can calculate $V^+(n, s)$ together with (overlapping) values of $W(n, s)$ in a single backward pass.

The relative simplicity of this model depends heavily on the assumption of path-independency of the prepayment function $\lambda(n, s, \alpha)$. In case of path-dependency in the parameter vector α a different procedure must be used. The best approach would probably be to move the specification of λ from the decision date to the settlement date, and let λ depend on lagged values of interest rates. Some simple forms of path dependencies may be handled by a variant of the forward induction method[40], while more complicated types of path dependencies require the use of Monte Carlo techniques, cf. chapter 7.

3.4 Summary

We have reviewed the arbitrage-free term structure models with special reference to the model proposed by Black, Derman and Toy(1990), and shown how certain interest rate dependent securities can be priced in a binomial lattice framework. Section 3.3 developed a MBB pricing model in which the prepayment function summarizes the behaviour of the individual mortgage holders. In chapter 5-7 it will be shown, how different models arise from different assumptions on the prepayment function.

40 This will be the subject of a forthcoming paper.

4 Arbitrage-Free Models with Taxation.

While the construction of an arbitrage-free lattice in a pre-tax setting is fairly standard and widely accepted, there exists to our knowledge no common way in which a pre-tax arbitrage-free model is extended into a model which allows for different tax schemes for different groups of investors. As discussed in later chapters this diversity of tax-schemes is nevertheless an empirical fact having a profound effect on the valuation of mortgage backed bonds.

In the current section we discuss this extension of the arbitrage-free lattice model. Section 4.1 reviews work by several authors showing that no bond market equilibrium exists with differential taxation unless some kinds of institutional restrictions are accepted. Section 4.2 discusses the valuation of fixed-payment after-tax cash flows. An institutional environment is proposed in which bonds are divided into three groups according to coupon rate and date of issue. In this environment a unique after-tax term structure will exist, which is used by private investors to value any type of fixed-payment cash-flow. In section 4.3 we show how to construct an arbitrage free lattice model, which matches the initial after-tax term structure as well as the pre-tax one. This model will be used in the remaining chapters for the after-tax calculations of options.

4.1 Differential taxation and no-arbitrage equilibrium

In the following analysis of tax systems in the Danish bond market investors are divided into two groups. The first group called the private investors are taxable of coupon income, but not of capital gains, while the second group - the institutional investors - are taxed of capital gains and coupon payments at the same rate.

Private investors include all households while the institutional investors consist of banks, public institutions, insurance companies, pension funds etc. The classification of corporations depends on the date of issue of their bonds. For bonds issued before 1991 the firms are taxed as private investors, while they belong to the institutional investors regarding any issue from 1991.

Using a single-period model Shaefer(1982) shows that the existence of different tax schemes opens arbitrage opportunities in a bond market. Consider for example two one-period bonds with coupon rates of 6% or 10% respectively, and let the tax rate for private investors be 50%. Irrespective of the before and after tax interest rates the institutional investors will accept relative prices of 106/110 while private investors set relative prices of 103/105. This opens a risk free arbitrage possibility, in which private investors could gain by issuing high coupon bonds using the proceeds to invest in low coupon bonds. Institutional investors would conversely sell low coupon bonds and invest in high coupon issues. The process would continue until net taxable income from the private investors became zero. Only by the introduction of short sale restrictions will the market reach a stable equilibrium.

Raaballe and Toft(1990) extend the analysis of Schaefer to a multi-period model. The one-period assumption of two bonds with different coupon rates is substituted by the so-called *double spanning condition* (DSC). In their first model trading only occurs at time zero, and the DSC requires that two bonds with different coupon rates exist at time zero for each maturity. A second model allows trading at each future date-event and the model becomes a series of one-period models, in which the DSC requires that two one-period bonds with different coupon exists at the start of each future date-event. Assuming a bond market no-arbitrage equilibrium with no restrictions on short sale, Raaballe and Toft show that any investor uses the same marginal tax rate for capital gains as well as coupon payments. The marginal tax rate may differ between different investors. For private investors the tax rate is zero, reflecting the fact that any taxable income at any future date could be removed by bond arbitrage between private and institutional investors.

The logic behind the models by Raaballe and Toft can be illustrated by a simple example. Consider a private investor who at the start of some future year learns that his taxable income will be DKK 200.000 and his tax rate 60%. To avoid paying DKK 120.000 in taxes he routinely acquires a DKK 11,000,000 loan from his bank (an institutional investor) at the one-period market rate of 12%. The proceeds are invested in nominal DKK 11,200,00 10% bonds. At the end of the year his 10% bonds are worth DKK 12,320,000 which exactly matches the amount due on his loan. The transaction will be entirely risk free for the institutional investor if the bonds are placed as collateral. The investor has paid interest of DKK 1,320,000 and received interest of DKK 1,120,000 plus a tax-free capital gain of 200,000. Deducting the net interest payments of DKK 200,000 leaves his total taxable income at zero. Knowing this to be possible at any future date he considers his marginal tax rate to be zero.

The models by Shaefer(1982) and Raaballe and Toft (1990) show how difficult it is to combine differential taxation with rational behaviour. Taken for granted the Raaballe and Toft model simply states that differential taxation cannot exist.

It would be tempting to accept their argument and continue using pre-tax valuations for all kinds of investors, but differential tax schemes is nevertheless an empirical reality. The mere fact that private households in Denmark contributed 206 billion DKK in 1990 in direct taxes which amount to 25.5% of total GDP indicates that there is room for some modifications of the model.

The most important modification is the so-called *minimum interest rule* which is in fact a maximum rule as well. This rule basicly forbids credit institutions to issue loans above par while any capital gain from bonds issued with coupon rates below the minimum interest rate will be fully taxed even for private investors. The minimum interest is continuously adjusted to keep newly issued bonds close to par. The minimum interest rule effectively reduces the range of available coupon rates.

For small differences in coupon rates a large volume of transactions is needed to obtain a specific tax-advantage. The private investors therefore incur increased transaction costs, which reduces the gains from arbitrage.

The minimum interest rule can be viewed as a short sale restriction of bonds issued above par contrary to the assumption of unrestricted short sales used in the model of Raaballe and Toft.

The minimum interest rule only applies to new bond issues. Private investors still has the opportunity of investing in earlier issued lower coupon bonds. Because most bonds are issued with coupons at or above current market rates lower coupon bonds are however in limited supply and they are easily absorbed by private investors[41]. As shown in Shaefer(1982) this leads to a segmented bond market, in which the prices of low coupon bonds are set by private investors based on their after tax cash flow, while bonds at or above the minimum rate are priced by institutional investors.

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[41] This could change, if interest rates rise above say 11%, which would leave an outstanding volume of nominal DKK 1000 billion low coupon bonds readily available for tax arbitrage. However, a temporary restriction on close-circuit loan arrangements has recently been put forward by the Danish Government.

To conclude the discussion of tax arbitrage we still feel a need for an inclusion of differential taxation in the bond pricing model. Tax arbitrage possibilities do exist but the institutional restrictions limit their availability, which explains why the majority of private investors probably use a myopic approach, valuing tax advantages only when they occur in a natural setting like mortgage finance. To reflect this view in the model our next section will clarify and strengthen the institutional restriction needed.

4.2 Pricing fixed-payment after-tax cash flows

An important application for after-tax discounting is the analysis of private mortgage prepayment, in which the old loans after-tax cash flow is compared to the after-tax cash flow of a new loan. In the current section we shall discuss after-tax valuation assuming cash-flows to be non-callable.

The pricing of non-callable after-tax cash flows can be done by a yield to maturity approach or by the introduction of an after-tax term structure. In the following τ denotes the tax rate on interest income. In the examples we assume a tax rate of 50%.

The after-tax yield to maturity approach is by far the most popular method, used in almost all counselling on private mortgage prepayment. In this method the after-tax value is found by discounting after-tax cash flows by the after-tax yield to maturity of a new loan. If the new loan is issued at par with annual interest payments, the after tax yield y^{τ} can be found as $y^{\tau} = (1-\tau) \cdot y$, where y denotes the pre-tax yield. If the new loan is issued at prices away from par, a numerical procedure is needed to solve for the after-tax yield to maturity[42].

The after-tax yield to maturity may be well suited in a highly regulated mortgage credit market, in which the pay schedules of the new loan is well defined. However,

[42] The conventional approach, used throughout this paper, assumes tax payments to occur at the settlement dates for interest payments. A 12% par loan with quarterly payments will thus have a pre-tax yield to maturity of 12.55%, while the after tax yield is $(1+(1-0.5)\cdot 0.03)^4 - 1 = 6,14\%$. A similar loan with an annual coupon of 12.55% would likewise be priced at par but now the after tax yield is 6.27%. As most households pay and deduct taxes independent of the exact timing of interest payments this implies a rather arbitrary distinction between loans with 1, 2 or 4 settlement dates per year. A more uniform but also more complicated approach would be to modify the pre-tax cash flow according to the settlement dates for tax payments.

with the current trend towards deregulation in the Danish mortgage market, borrowers have access to loans of different maturities, and we shall therefore use an after-tax term structure approach similar to the pre-tax model.

From the discussion of tax-arbitrage possibilities in the previous section it is evident that some very specific assumptions concerning the institutional environment are needed. We assume a strict version of the minimum interest rule. At any date the government divides new bond issues into A- and B-bonds[43]. A-bonds exists for all maturities, but only one coupon rate and one loan type is allowed for each maturity. All other bond issues are labeled B-Bonds. Private investors can issue and purchase A-bonds of any maturity being taxed of coupon payments only. Private investors are not allowed to issue B-bonds and capital gains on B-bonds are fully taxed even for private investors. Capital losses on B-bonds are not tax-deductable. A third group labelled C-bonds consists of earlier issued A-bonds. Capital gains on C-bonds are tax-exempt, but the outstanding volume of these bonds is assumed to be small compared with total taxable income from private investors. To simplify computations it is finally assumed that A-bonds are issued as ordinary loans, although it would be simple to use annuities or serial bonds instead.

The institutional investors are free to issue and invest in A-bonds as well as B-bonds. We finally assume that all investors only considers the tax arbitrage possibilities at time zero.

A formal analysis of the equilibrium properties of this model would be highly appropriate, but is outside the scope of the current thesis. The following discussion should therefore be seen as a preliminary conjecture on the properties of a no-arbitrage equilibrium.

As private investors have no short sale restrictions on A-bonds and institutional investors have no restrictions on either A- or B-bonds then the no-arbitrage condition implies that prices of these bonds are set according to a pre-tax bond arbitrage equilibrium that is

[43] A similar distinction exists in the Danish minimum interest rate law discussed above. The law refers to A-bonds as ’blue-stamped’, while B-bonds are ’black-stamped’. These terms are inherited from the agricultural sector referring to either excellent or average quality of Danish bacon.

(4.1) $$P = \sum_{t=1}^{M} d_t \cdot (f_t + c_t)$$

where d_t denotes the pre-tax discount factors while f_t denotes repayments on principal and c_t coupon payments on the bond.

Let c_n be the coupon rate of an ordinary A-bond maturing in period n having a face value of DKK 1. The price is given by

(4.2) $$P_n = c_n \sum_{t=1}^{n} d_t + d_n$$

The precise structure of A-coupon rates is not crucial for the model, but for completeness one could assume A-bonds to be issued at par, i.e. $P_n \equiv 1$, which leads to the following recursive expression for the term structure of A-bond coupon rates:

(4.3) $$c_n = (1 - d_n) / \sum_{t=1}^{n} d_t$$

The private investors price A-bonds according to the after-tax cash-flow, i.e.

(4.4) $$P_n^{\tau} = (1-\tau) c_n \sum_{t=1}^{n} d_t^{\tau} + d_n^{\tau}$$

where d_t^{τ} denotes the after tax discount factor for time t. With no short-sale restrictions on A-bonds, $P_n^{\tau} = P_n$, and d_t^{τ} can be found by:

(4.5) $$d_n^{\tau} = \frac{P_n - (1-\tau) c_n \sum_{t=1}^{n-1} d_t^{\tau}}{1 + (1-\tau) c_n} \quad \text{for } n = 1, 2, ..$$

The after-tax discount factors are uniquely determined by the allowed refinancing alternative and they reflect the relative weight of interest payments in the bonds as well as the current shape of the term structure.

To argue that a bond market equilibrium exists, it is necessary to go through the different possibilities.

By trading in newly issued A-bonds the private investors are able to replicate any after-tax cash flow from a C-bond i.e. an earlier issued A-bond. The value of a C-bond for private investors are given by

(4.6) $$P^{\tau} = \sum_t d_t^{\tau}(f_t + (1-\tau)c_t)$$

This after-tax value will in general differ from the pre-tax value given by relation (4.1). For P^{τ} greater than the pre-tax value P the bond will be held by private investors and its market price will be equal to P^{τ}. In equilibrium the short-sale restriction for private investors will be binding for these bonds. If P^{τ} is less than the pre-tax value P, the bond will be held by institutional investors. Trading continues until all low coupon C-bonds ($P^{\tau} > P$) are held by private investors while the corresponding high coupon bonds ($P^{\tau} < P$) are held by institutional investors[44].

Consider now the private investors valuation of B-bonds. If the coupon rate on a B-bond is above the current A-bond rate, then the after-tax value will be below the pre-tax value, which means that the bond should be held by institutional investors. If the coupon rate on the B-bond is below current A-bond rates, then capital gains are fully taxed for private investors. This means that the after-tax value is either less than or equal to the pre-tax value depending on wether current A-bonds contain any tax exempt capital gain. Either way the B-bonds are unattractive for private investors, while the short sale restriction prohibits any tax arbitrage. B-bonds are thus held and priced by institutional investors alone.

The ABC-model can be seen as full term structure version of the well-known after-tax yield to maturity approach. The after-tax yield to maturity of the new loan is used, because it represents the allowed refinancing alternative for a private investor, i.e. the A-bond. Pricing an earlier issued higher-coupon mortgage with the after-tax yield on the new loan reveals that the mortgage holder has a tax-advantage, compared to the

[44] The structure of bond market prices becomes less clear-cut, if the recent temporary restriction of close-circuit loan arrangements mentioned above is included in the model. With short-sale restrictions even on A-bonds, no uniquely determined after-tax term structure exists in general and private investors may no longer be able to absorb the outstanding volume of lower coupon C-bonds. A linear programming approach like the one in Schaefer(1982) might be used instead, but this must await further research.

refinancing alternative[45]. If private investors could freely issue new higher-coupon loans this tax-advantage would disappear, but so would the bond market no-arbitrage equilibrium.

The ABC-model may also be compared with the model by Raaballe and Toft(1990). The double spanning condition is fulfilled in our model, because a whole range of C-bonds with different coupon rates exist in the market. There exist nevertheless a no-arbitrage equilibrium with differential taxation because the supply of earlier bond issues is restricted. In the Raaballe and Toft model private investor could freely choose a position with deductable interest payments being matched by non-taxable capital gains. This is not possible in the ABC-model, because unlimited short-sale applies to only one coupon rate and one bond type for each maturity. Purist could say that private investors face an incomplete market, because coupon payments cannot be disentangled and traded separately, but the Minister of Finance is probably happy with the arrangement above.

While the present model sets up a viable bond market no-arbitrage equilibrium with differential taxation it is evident that small changes in the assumption could offset the equilibrium. Allowing even a narrow range of A-bond coupon rates for the same maturities will bring up the situation described by Raaballe and Toft (1990) in which private investors face an effective tax rate of zero. If market rates increase, then the supply of earlier issued lower coupon A-bonds increases, and arbitrage arrangements may no longer be in short supply relative to taxable income for private investors. Finally we have assumed that investors considers only tax arbitrage possibilities at time zero.

4.3 A stochastic after-tax model

In the previous section we modelled an arbitrage-free after-tax bond market equilibrium with private as well as institutional investors. To avoid tax-arbitrage a number of restrictive assumptions was made. Private investors were only allowed to issue A-bonds, capital gain from B-bonds were fully taxed even for private investors, the supply of C-bond was limited and all parties implicitly followed a buy-and-hold strategy considering only tax arbitrage possibilities at time zero. Finally all bonds were assumed to be non-callable.

[45] As shown in chapter 5 this tax-advantage may reduce the borrowers gain from prepayment.

In this section the model is extended to allow for the simultaneous valuation of state-dependent pre-tax and after-tax cash flows. The model will later be used for the analysis of the prepayment option embedded in callable mortgage backed bonds, but a similar model would be needed for the analysis of all kinds of interest rate options traded in markets with differential taxation.

In Christensen (1985) a model for the valuation of callable mortgage backed bonds is studied, in which the bonds are issued by private investors and bought by institutional investors. He uses the one factor continuous time stochastic model developed in Cox, Ingersoll and Ross(1985), but the main assumptions concerning after-tax valuation can be translated to the binomial arbitrage-free model used in the present paper.

Christensen assumes a pre-tax arbitrage-free bond equilibrium. In the present context this corresponds to the binomial arbitrage-free model described in section 3.1. The callable mortgage held by the private investor is divided into a non-callable mortgage plus a prepayment option allowing the private investor to buy back his non-callable loan at face value. The after-tax value of this option is determined by the following procedure:

At each date-event the pre-tax model is used to find the pre-tax value and yield to maturity of the non-callable mortgage. Assuming new loans to be issued at par the after-tax value of the non-callable mortgage can be found by discounting after-tax cash flows with the after-tax yield to maturity of a new loan. We shall refer to this as the par yield to maturity (PYTM) method. Subtracting the face value of the mortgage provides an after tax estimate of the exercise value of the option at each date-event.

To discount exercise values down to time zero Christensen extends the pre-tax stochastic model by assuming the short rate to be fully tax deductable. This leads to a partial differential equation for the after tax value of the option[46]. In the binomial setting we get the following after-tax version of the backward equation:

$$
\begin{aligned}
V^{\tau}(n,s) &= f(n,s) + (1-\tau)c(n,s) \\
&\quad + p^{\tau}(n,s)\,\{\theta(n,s)\cdot V^{\tau}(n+1,s+1) + (1-\theta(n,s))\cdot V^{\tau}(n+1,s)\}
\end{aligned}
\tag{4.7}
$$

[46] Cf. Christensen (1985) relation (32).

where the superscript τ indicates that all interest payments in the cash flow have been converted to after tax payments by the multiplication of $(1-\tau)$. $p^{\tau}(n,s) = \exp(-r(n,s)(1-\tau)dt)$ denotes the one-period after-tax discount factor. We shall refer to this as the *local no-arbitrage* method (LNA).

The valuation procedure is attractive in the sense that the widely used after-tax yield to maturity method is used for the main non-callable part of the mortgage, while the stochastic differential equation only applies to the option part, the valuation of which is less clear-cut anyway.

One drawback of the method is the heavy computational costs. For the valuation of a 30 year mortgage in a lattice with monthly time steps close to 65,000 different date-events must be considered. Assuming quarterly payments a third of these date-events will be decision points for the prepayment option. At each decision point a numerical calculation of pre-tax yield to maturity is necessary and must be followed by the discounting of the remaining after-tax cash flow from the mortgage. Even if a fast computer could do the job in a few minutes the practical use of the model would be limited.

Our second objection is of a more theoretical nature. Christensen (1985) employs two after-tax valuation methods, the par yield to maturity (PYTM) and the local no-arbitrage method (LNA). But in general these methods give different results.

The local no-arbitrage method can be used to price non-callable cash flows as well. To compare the methods we have therefore calculated after-tax values of different non-callable bonds.

Table 4.1 illustrates the percentage difference between after-tax and pre-tax value of different 20 year annuity bonds with quarterly payments. Only after-tax values varies across columns. Calculations are done for coupon rates ranging between 5 and 20%. Three different choices have been made for the initial term structure: A linearly declining yield curve starting at 15% with a yearly slope of -0.2273%, a flat yield curve at 12.55% and a linearly rising yield curve starting at 9% with a slope of 0.3552%. All three yield curves have been carefully selected to price the 12% coupon bond at par.

The first column contains the par yield to maturity (PYTM) values. By the annuity assumptions all bonds share the same pre-tax yield to maturity, which in addition is the same for all three yield curves. The 12% bond is priced at par and its pre-tax value

Table 4.1: *Difference between after-tax and pre-tax values as a percentage of pre-tax value for different coupon rates.*

	Par yield to	Local no-arbitrage method			Par value arbitrage-free		
Coupon	maturity	Declining	Flat	Rising	Declining	Flat	Rising
5.0	20.72	21.14	18.62	14.45	20.87	20.95	21.00
6.0	16.48	16.95	14.42	10.26	16.68	16.71	16.69
7.0	12.78	13.29	10.76	6.60	13.02	13.00	12.92
8.0	9.54	10.08	7.55	3.41	9.81	9.75	9.63
9.0	6.69	7.25	4.73	0.61	6.99	6.90	6.73
10.0	4.18	4.76	2.24	-1.85	4.50	4.38	4.19
11.0	1.96	2.56	0.05	-4.01	2.30	2.16	1.94
12.0	0.00	0.61	-1.88	-5.91	0.35	0.20	-0.04
13.0	-1.74	-1.12	-3.60	-7.59	-1.38	-1.55	-1.80
14.0	-3.30	-2.67	-5.13	-9.09	-2.92	-3.10	-3.36
15.0	-4.68	-4.05	-6.49	-10.41	-4.30	-4.49	-4.76
16.0	-5.92	-5.29	-7.71	-11.60	-5.54	-5.73	-6.00
17.0	-7.03	-6.40	-8.80	-12.65	-6.65	-6.84	-7.12
18.0	-8.03	-7.40	-9.79	-13.60	-7.65	-7.85	-8.12
19.0	-8.93	-8.31	-10.67	-14.45	-8.56	-8.75	-9.02
20.0	-9.75	-9.13	-11.47	-15.21	-9.37	-9.57	-9.84

coincides with its after-tax value. Bonds with coupon rates below 12% is valuable to a private investor compared to the par bond, while bonds with coupon rates above 12% is attractive for private borrowing. As discussed in the previous sections there is room for infinite tax arbitrage, if private investors were freely allowed to issue high coupon bonds and invest the proceeds in low coupon bonds.

The next three columns shows the same calculation using the local no-arbitrage method (LNA). The results differs sharply from par yield to maturity values. For the 12% par bond the LNA method results in deviations of after-tax values ranging between 0.61% and -5.91% depending on the yield curve. Similar results apply to the other coupon rates. The different after-tax values stem from differences in discounting. The PYTM method uses the same after-tax discounting rate throughout the loan. The LNA method discount at each date-event using a range of one-period after-tax rates. In general LNA seems to underestimate after-tax values relative to PYTM for flat and rising yield curves, while the results for the declining yield curve correspond rather closely.

To summarize it was shown that the LNA method is unable to fit the after-tax values computed by the PYTM method. On the other hand the PYTM method is computationally demanding and it can only be used on non-callable cash flows.

In the following we shall suggest a slight modification of the LNA method. The local arbitrage condition (4.7) still applies, but private investors use a different set of martingale probabilities, which are consistent with the assumed structure of after-tax discount factors at time zero.

The column labeled "par value arbitrage-free" (PAF) contains the values of an after-tax lattice in which the future martingale probabilities are adjusted in such a way that the after-tax calculation corresponds to the current after-tax term structure. In the example the after-tax term structure is derived by assuming that the current A-bonds consists of ordinary bonds issued at par, cf. section 4.2. As seen, the values are very close to the par yield values. The small difference in the valuation of the 12% par bond is due to the fact that the after-tax term structure is computed from ordinary bonds, while the PYTM valuation is based on a 12% annuity bond.

The after-tax lattice can be very efficiently constructed as a special case of the forward induction method introduced by Jamshidian(1991) and discussed in section 3.1.4. The first step is to calibrate the pre-tax lattice. Any version of the arbitrage-free model could be used. The calibration procedure results in a lattice of future short rates, $r(n,s)$. To derive the after tax lattice we assume the following:

1) Private investors agree with institutional investors on the lattice of future short term interest rates $r(n,s)$.

2) Short-term interest rates are fully taxed at the rate τ.

3) Private investors use a time dependent martingale probability θ_n^τ.

4) The after-tax valuation of private investors matches the initial after-tax term structure $d^\tau(t)$.

As the martingale probabilities are closely connected to the risk preferences of the investor it seems reasonable to assume that these values differs between private and institutional investors. The value of θ_n^τ can be derived by the following procedure:

Let $p(n,s) \equiv \exp(-r(n,s)(1-\tau)dt)$ denote the one-period after-tax discount factor at date-event (n,s), while $d^\tau(n)$ denotes the n-period after-tax discount factor at time zero. $d^\tau(n)$ could be found from the term structure of A-bonds as shown in the previous section. Finally we let $G(n,s)$ denote the after-tax Arrow-Debreu prices i.e. the value at time zero of a primitive security having an after-tax cash flow of one at date-event (n,s) and an after-tax cash flow of zero elsewhere. Defining $G(n,-1) \equiv G(n,n+1) \equiv 0$ and $p(n,-1) \equiv p(n,n+1) \equiv 0$ we can write the forward equation (cf. section 3.1.4) as

$$G(n+1,s) = \theta_n^\tau p(n,s-1)G(n,s-1) + (1-\theta_n^\tau)p(n,s)G(n,s) \tag{4.8}$$

The Arrow-Debreu prices can be used to value any kind of time- and state-dependent after-tax cash flow. As a special case we have

$$d^\tau(n+2) = \sum_{s=0}^{n+1} G(n+1,s)p(n+1,s) \quad . \tag{4.9}$$

To interpret equation (4.9) consider a security paying an after-tax cash flow of one at time n+2 independent of state. At date-event $(n+1,s)$ the security will be worth $p(n+1,s)$. Its value at time zero is found as the summation across all states at time $n+1$ using the Arrow-Debreu prices $G(n+1,s)$ and by assumption 4) this value must be equal to $d^\tau(n+2)$.

Substituting $G(n+1,s)$ from equation (4.8) gives the following relation between $d^\tau(n+2)$ and θ_n^τ

$$\begin{aligned} d^\tau(n+2) &= \theta_n^\tau \sum_{s=0}^{n+1} p(n,s-1)G(n,s-1)p(n+1,s) \\ &\quad + (1-\theta_n^\tau) \sum_{s=0}^{n+1} p(n,s)G(n,s)p(n+1,s) \\ &\equiv \theta_n^\tau S_n^- + (1-\theta_n^\tau)S_n^+ \quad . \end{aligned} \tag{4.10}$$

Equation (4.10) defines $d^{\tau}(n+2)$ as a linear function of θ_n^{τ}. Solving for θ_n^{τ} a recursive relationship is found, which determines the after-tax martingale probabilities, θ_n^{τ}, as a function of $G(n,s)$, $p(n,s)$ and $p(n+1,s)$:

$$(4.11) \qquad \theta_n^{\tau} = \frac{d^{\tau}(n+2) - S_n^{+}}{S_n^{-} - S_n^{+}}$$

From a numerical point of view relation (4.11) provides a very efficient algorithm. While a Newton-Raphson procedure was needed to calibrate the pre-tax lattice the fact that θ_n^{τ} enters linearly into relation (4.11) allows for a direct computation.

The after-tax lattice prices current A-bonds correctly, but the procedure does not assure that after-tax valuation of future A-bonds will fulfil the same condition. This could be a problem, especially if the lattice is used for the evaluation of complex dynamic tax-arbitrage strategies[47].

4.4 Summary

In this section we have discussed differential taxation and bond market equilibrium. As shown by several authors a no-arbitrage equilibrium with differential taxation cannot exist without restricitions on either trading possibilities or rationality.

We have suggested an ABC-model for the bondmarket, having an institutional setting close to the current Danish tax system. The main restriction is that only A-bonds, i.e. one specific bond type and only one coupon rate, could be freely issued by private investors. No-arbitrage equilibrium for A-bonds ensures that pre-tax and after-tax values coincide, which uniquely determines an after-tax term structure. The no-arbitrage equilibrium is characterized by a segmented bond market with earlier issued low coupon bonds being held solely by private investors. As long as the supply of earlier issued low-coupon bonds is limited relative to gross taxable income, differential taxation exists with private investors valuing cash flows on an after-tax basis.

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[47] As an alternative route of research it may be possible to point out a single bond at each date-event as the relevant refinancing bond, say a par bond, and set risk neutral probabilities in order to price it at the same price before and after tax.

In the last section taxes was introduced into the intertemporal arbitrage-free model. It was shown that by assuming different risk preferences between private and institutional investors, the same model could be consistent with the initial pre-tax as well as the initial after-tax term structure. The after-tax lattice was developed by an efficient algorithm based on the Jamshidian(1991) forward induction technique.

5 Optimal Prepayment Behaviour

The individual borrower can be seen as a holder of a non-callable mortgage plus an American option, giving him the right to buy back the mortgage at face value on any date until maturity. It would thus seem natural to model the prepayment behaviour along the lines of standard option theory, in which case the borrower follows an optimal value-minimizing strategy, calling his loan as soon as the value of the non-callable mortgage less the value of the option exceeds the call price.

Models with this kind of borrower behaviour has been proposed and analysed by e.g. Brennan and Schwartz (1977) in the case of standard callable bonds while Christensen(1985), Jakobsen and Tanggaard(1986), Mouritsen and Møller(1987) and Dahl(1991) analyses Danish MBBs. This chapter will study the American option model for MBBs in the framework of the binomial model developed in section 3.3. The American option model will provide insight into the decision problem facing the individual borrower, but as will become evident the model has some limitations when used as a pricing model for MBBs.

Section 5.1 specifies the prepayment function of the American option (AO) model. The prepayment function is determined by optimal prepayment behaviour and section 5.2 shows how rational borrowers would react to changes in the economic environment. Section 5.3 uses the after-tax framework of section 4.3 and shows that the introduction of taxes works similar, but not equivalent, to higher prepayment costs in a pre-tax model. Changing to a higher coupon rate for tax-reasons is possible in the Danish mortgage system and section 5.4 discusses how this ’delivery-option’ may affect prices through its impact on prepayment behaviour. We finally show, how severe discontinuity problems arise, when the prepayment function of the American option model is used in a pricing model for MBBs.

5.1 The basic decision problem

Assume for the moment that borrowers as well as investors value cash flows in a pre-tax setting[48]. In the absence of risk free arbitrage possibilities both parties will thus evaluate any interest rate dependent cash flow according to (3.1). In the American option model the borrower has to decide in each period between prepaying or continuing the loan. If he decides to prepay the loan, he has to pay the remaining principal at the first possible settlement date, plus any interest and repayments due in the period from his decision up to the final payment. The value of this is $W(n,s)$ as discussed in section 3.3. The borrower will furthermore encounter various transaction costs which are assumed to be a fixed percentage γ of $W(n,s)$. Total prepaying cost will thus be $W_m(n,s,\gamma) \equiv (1+\gamma)W(n,s)$. The value of the mortgage in case of no prepayment will be denoted $V_m^+(n,s,\gamma)$. Using the value-minimizing principle the value of the mortgage at date-event (n,s) is given by

(5.1) $$V_m(n,s,\gamma) \equiv \min\{W_m(n,s,\gamma), V_m^+(n,s,\gamma)\}$$

where by the standard pricing formula (3.20):

(5.2) $$V_m^+(n,s,\gamma) = f(n,s) + c(n,s) + \frac{1}{2} p(n,s)\{V_m(n+1,s+1,\gamma) + V_m(n+1,s,\gamma)\}$$

and $V_m^+(n_M,s,\gamma) = 0$ for all states s.

Under the assumption of rational value minimizing behaviour from the pool of mortgage holders the prepayment function is given by:

(5.3) $$\lambda(n,s,\gamma) = \begin{cases} 1 & \text{if } W_m(n,s,\gamma) < V_m^+(n,s,\gamma) \\ 0 & \text{otherwise} \end{cases}$$

The price of a MBB can now be found by a standard backtracking procedure. At each date-event starting from maturity one first calculates the value of W_m and W_m^+ in order to evaluate the prepayment function and then uses the prepayment function to calculate the value of the MBB by formula (3.19) . Repeating this two-step procedure down to date-event (0,0) completes the pricing model.

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48 The model of this section will be a close to Dahl(1991). The paper by Dahl concentrates on the empirical results, but according to the paper, he uses a pre-tax BDT-model with a constant cost-rate. The term of notice is not modelled explicitly. The current chapter can be seen as a more formal statement and analysis of Dahls model with several extensions.

5.2 Determinants of critical yield

In this section the borrowers prepayment decision will be studied in more detail. As no closed form expressions are available for this kind of model the argument will be supported mainly by numerical examples. Many of the observations have been stated elsewhere, but we summarize them here for easy reference. Unless otherwise stated the basic parameters for the examples are the following. Valuation is done using the BDT-model as described in section 3.1 with 32 steps per year[49] and a constant future short rate volatility of 15%[50]. Yields are quoted using annual compounding and we assume the initial term structure to be flat. Mortgages are annuities with 20 years remaining, quarterly payments and an annual coupon of 12%. The cost rate γ as well as the term of notice will initially be set to zero. To ease comparison market prices and mortgage values are shown net of accrued interest.

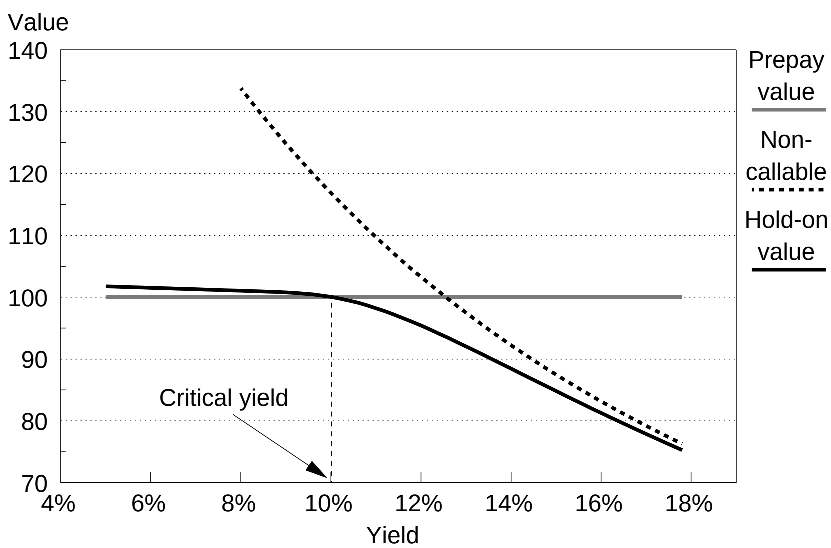

*Figure **5.1**: The comparison between prepayment and hold-on value of the mortgage.*

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[49] Computations of critical yield use only 8 steps per year.

[50] A two stage PC program developed by the author has been used in the following chapters. The first step calibrates pre-tax and after-tax lattices given any specification of the initial term structure and volatility curve. The second step performs the backwards pricing procedure for the different pre-tax and after-tax versions of the American option model. The calculations of each critical yield reported below are done with several iterations of the calibration and valuation steps using a Newton-Raphson procedure. A full summary of pricing results is saved in the Paradox database format at the end of each valuation. The databases are later crosstabulated and used as input to various graphing programs etc.

The program can also be used with the required gain model of the next chapter. For the empirical analysis of chapter 7 these calibration and valuation procedures have been transferred to another PC program, RIO/Optikon, developed by the author, which uses actual yield curve estimates and bond information to derive price and duration estimates for different samples of Danish MBBs.

The basic prepayment behaviour is illustrated in figure 5.1 for different yield levels. For each date-event (n, s) and given the initial term structure the borrower is assumed to compute the prepay value of the mortgage, W_m. This is compared to the hold-on value of the mortgage, V_m^+. If $V_m^+ > W_m$ the optimal strategy will be to prepay the loan, while the rational borrower should continue his loan in case $V_m^+ < W_m$. The *critical yield*, r^*, is defined as the yield level, at which the two curves intersect. If the term structure is flat the critical yield coincides with the common level of interest rates, while in general the critical yield is defined as the yield to maturity of a bond with payments equal to the MBB assuming no prepayments. Analogous to earlier work see e.g. Christensen(1985) the critical yield is used as a convenient statistic summarizing the prepayment potential of the individual mortgages.

The critical yield is a sufficient statistic as well due to the fact that a knowledge of critical yield for all future dates provide full knowledge of the stochastic cash-flow to the investors. Different models for prepayment behaviour, with equivalent values of critical yield for all future dates, should therefore give rise to equivalent prices for the MBB. This possibility will be discussed in the next section.

It can be illuminating to split the value of the mortgage into the value of the underlying non-callable mortgage, B_m, and the value of the prepayment option, C_m, with $C_m = B_m - V_m$. Likewise we can at any time define the exercise value of the prepayment option as $C_m^* \equiv \max\{0, B_m - W_m\}$ and the hold-on value of the option as $C_m^+ \equiv \max\{0, B_m - V_m^+\}$. From the option point of view the mortgagor should prepay, when the exercise value exceeds the hold-on value that is $C_m^+ \leq C_m^*$. The two option-values are shown in figure 5.2. In general both the value of C_m^+ and C_m^* will be an increasing function of B_m and thereby a decreasing function of the level of interest rates[51].

The borrower is allowed to prepay at any date, but the rational borrower would postpone the decision as long as possible that is to the decision dates, ν_k, defined in section 3.3. If he should decide to prepay between the dates ν_{k-1} to ν_k, his debt at date ν_k will be equal to $W_m(\nu_k)$. If the prepayment decision is deferred to date ν_k, he has the option of continuing the loan, leaving the value of his mortgage at

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[51] As with almost any conclusion drawn in this paper there will probably exist cases in which the opposite is true, especially if extreme volatility structures are permitted. The conclusion refers to the "normal" case represented by the numerical examples, which hopefully covers the cases of empirical interest.

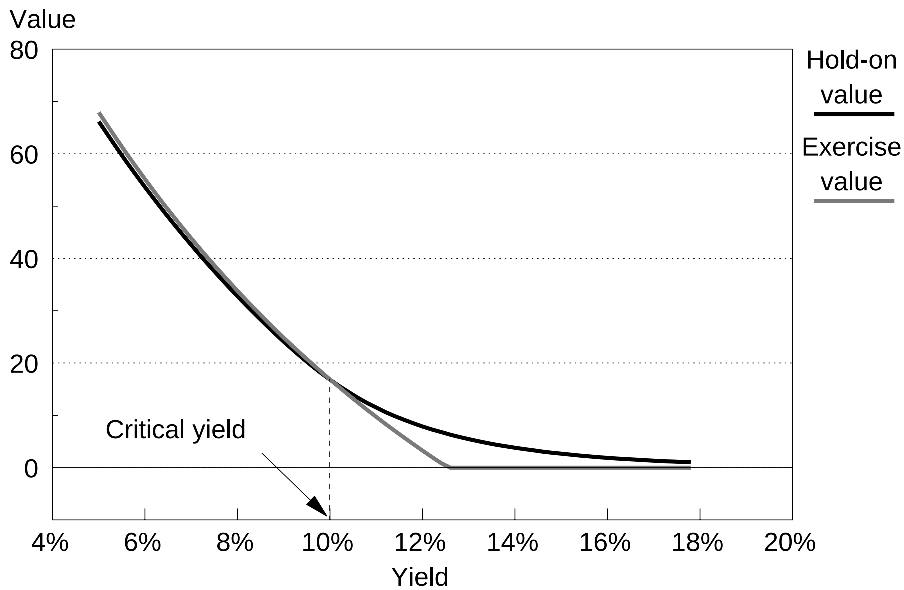

Figure 5.2: Comparison between exercise and hold-on value of the borrowers prepayment option.

$V_m(\nu_k) = \min(W_m(\nu_k), V_m^+(\nu_k))$. As any cash-flow due up to ν_k will be unaffected by the prepayment decision, we can conclude that $V_m^+ \le W_m$ at any date between ν_{k-1} and ν_k. In the remainder of this chapter the prepayment evaluation is therefore done for decision dates only.

We start by showing how time to maturity affects the prepayment decision. Assume the rate of transaction cost to be zero. At the last decision date before maturity the time value of the option is zero, and the borrower will prepay, if the mortgage value exceeds par. With a flat yield curve this corresponds to a critical yield equal to the *effective coupon rate*, defined by $\bar{r} = (1 + c/k)^k - 1$, where c is the coupon rate and k is the number of payments per year. At earlier decision dates, the borrowers take the time value of the option into account, which means that prepayment occurs at interest rates below the effective coupon rate i.e. $r^* < \bar{r}$.

The connection between time to maturity and critical yield is illustrated in figure 5.3 for different levels of the coupon rate[52]. As expected the critical yield is negatively related to the time to maturity, and it converges towards a constant long run level more than 2 percentage points below the effective coupon rate. The critical interest rate

52 The maximum length of Danish mortgages is 30 years, but in plots of critical yield as a function of time to maturity we use a maximum of 50 years to ease the comparison of asymptotic behaviour.

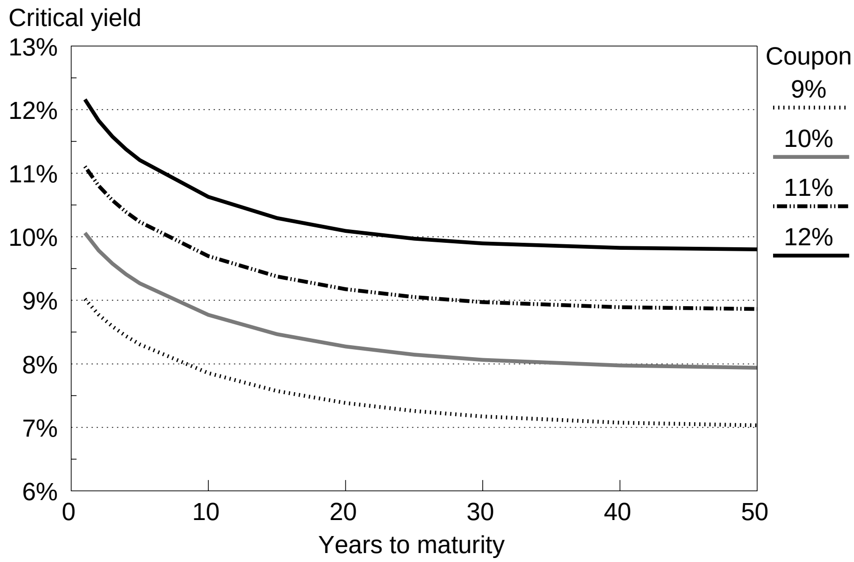

Figure 5.3: Connection between critical yield and time to maturity for different levels of the coupon rate.

depends on the coupon rate in a nearly linear fashion, indicating that loans with different coupon rates can be analysed simply by translating the results for a single coupon rate to a different yield level.

An increase in the cost rate, which is similar to an increase in the exercise price of the option, reduces C_m^+ as well as C_m^*. The direct influence on C_m^* is the strongest, leading to a fall in the critical interest rate, as shown by 5.4. The largest effect is seen for the shorter maturities due to the lower volatility of the underlying non-callable mortgage. As time to maturity increases, the underlying non-callable mortgage converges toward a constant volatility and thereby a constant impact from the cost rate. The term of notice would play a role similar to the cost rate, in that a longer term of notice increases the prepayment value, provided the current yield level is below the effective coupon rate.

In practice direct costs of prepayment lies somewhere between 2-5% depending on the size of the mortgage. Calculating with cost-rates of 20% thus seems grossly exaggerated. But as shown in the next section, taxation affects the prepayment behaviour of borrower in a way, which may be described as a very high level of prepayment costs.

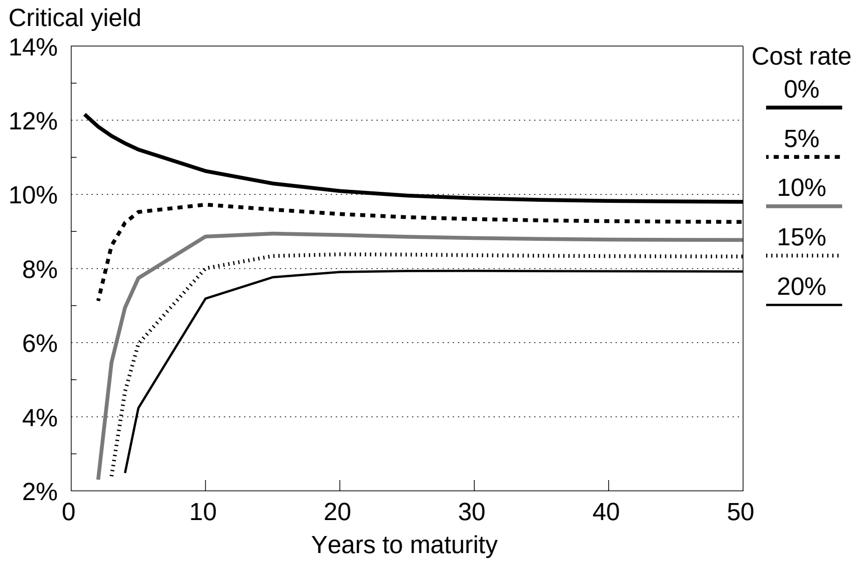

Figure 5.4: Comparison of critical yield and time to maturity for different values of the cost rate

5.3 After-tax prepayment behaviour

The preceeding discussion assumed that borrowers as well as investors valuate in a pre-tax setting. A more realistic assumption would be to assume that borrowers and investors decide on the basis of the after-tax cash flows associated with prepayment. The problem is that the two sides of the market represents two different tax systems. The representative mortgage owner, a firm or a private household, has an asymmetrical taxation scheme in that the interest payments are tax deductible while capital gains and losses are tax exempt[53]. On the other hand the typical bond investor that is a pension fund, an insurance company, a bank or a firm is equally taxed of interest payments and capital gains. Referring to the analysis of section 4 we shall refer to the two groups as private investors and institutional investors respectively.

To introduce differential taxation in the prepayment model one must modify the principle of arbitrage free pricing and accept some kind of market restrictions. Using the ABC-model introduced in section 4.2 we assume that private as well as institutional

[53] Starting from 1991 companies are taxed equally of interest income and realized capital gains. Corporate borrowing represents a borderline case in that capital losses stemming from issuing bonds below par is tax deductable if the coupon rate on the loan is below a certain minimum rate, at the present 8% p.a.. While the change in tax rules has opened up a market for low or zero coupon bonds for corporate borrowing this market is tiny compared to the amount of outstanding corporate loans with coupon rates at or above 8% for which only interest payments are tax deductable.

investors have unrestricted access to A-bonds of all maturities, but to avoid tax-arbitrage only one coupon rate is allowed for each maturity. This assumption uniquely determines pre-tax and after-tax discount functions, which is used to price non-callable cash flows. Then the par value arbitrage-free principle, described in section 4.3, is used to derive an after-tax lattice consistent with the current after-tax term structure. With these prerequisites pre-tax as well as after-tax values can be found for all interest rate dependent options.

Figure 5.5 shows the influence of the taxrate on critical yield for different times to maturity. With a term of notice equal to zero the prepayment value of the mortgage is independent of the tax-rate. When interest rates fall, the borrower has a mortgage with an effective coupon rate higher than prevailing market rates. With prices set by institutional investors high coupon loans are preferred relative to new loans with lower coupon rates. This inherent tax advantage lowers the hold-on value of the mortgage relative to the pre-tax situation, which eventually leads to a fall in the critical yield level. The critical yield for long maturities are affected most, because these mortgages have the largest difference between pre-tax critical yield and the coupon rate, and because their tax-advantage runs for a longer period. For very short maturities the pre-tax critical yield level is very close to the coupon rate, and the influence from taxation is minor.

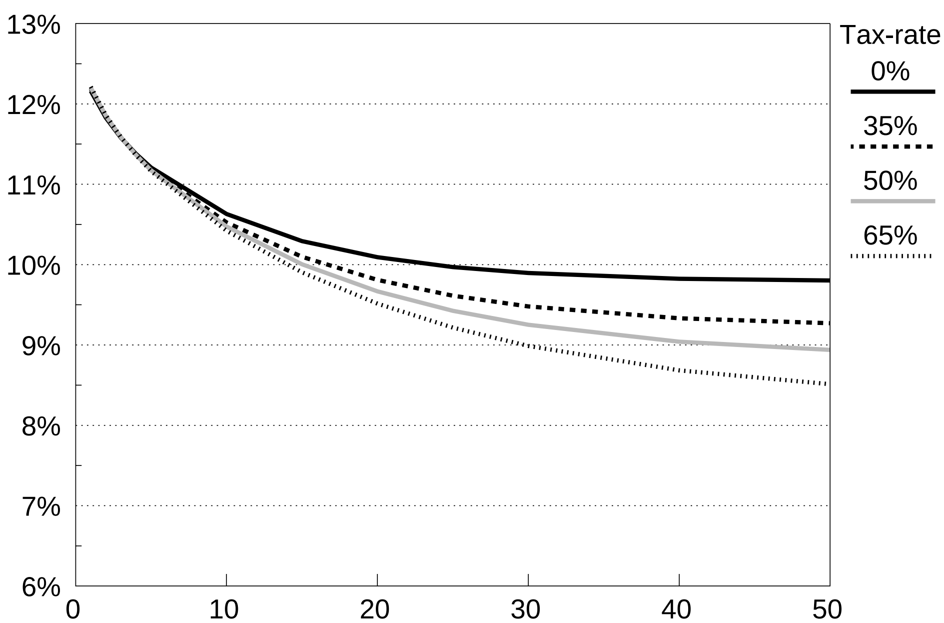

Figure 5.5: The influence of the tax-rate on critical yield for a 20 year 12% annuity mortgage with quarterly payments.

The impact from taxation is especially important for the so-called *cash-loans* issued prior to 1986. For these loans the borrower is allowed to deduct the full market rate of interest at the time of issuance. From 1986 all loans through credit institutions are issued as so-called *bond-loans*, for which only the coupon rate on the bonds is tax-deductable. Cash-loans issued below par therefore carries a tax advantage relative to an otherwise identical bond-loan. The tax-advantage is lost in case of prepayment, because the borrower has to repay the full face value of the bonds backing the loan.

Bonds issued during the cash-loan system are typically 20 or 30 year annuity bonds with semiannual payments and 10 or 12% coupon.

The cash-loan system can be illustrated by a simple one-period example: To finance a small mortgage a credit institution sells nominal DKK 100.000, one year, 10% bonds at a market rate of say 13%. During the cash-loan system the credit institution could now offer the borrower a loan with face value equal to the proceeds from sale that is 110,000/1.13 = DKK 97.345 and a coupon rate equal to the yield to maturity of the bond that is 13%. In this way the non-deductable capital loss on the bonds are converted to a fully tax-deductable interest payment.

Assuming taxation at 50%, the borrowers after-tax payments on the cash-loan would be $97,345 \cdot (1+(1-0.5)0.13) = 103,672$. After-tax payments on a similar bond-loan would amount to $100,000 \cdot (1+(1-0.5) \cdot 0.10) = 105,000$. As long as the cash-loan rate is higher than the coupon rate, the cash-loan has a tax advantage compared to a similar bond-loan. Note that after-tax payments on a bond-loan is equal to after-tax payments on a cash-loan issued at par.

In case of prepayment the borrower must pay the full DKK 100,000 face value of the underlying bond. For a one-year bond-loan prepayment would thus be optimal as soon as market rates drops below the coupon rate of 10%. For the one-year cash-loan with cash-rate of 13% prepayment would however not be optimal unless market rates drop below 7,34% corresponding to an after-tax payment on the new loan of 103,672.

To asses how the cash-loan system affects prepayment behaviour, we have computed critical yields of the standard 12% annuity bond with quarterly payments for different values of the cash-loan rate and for different values of time to maturity. The results are summarized in figure 5.6.

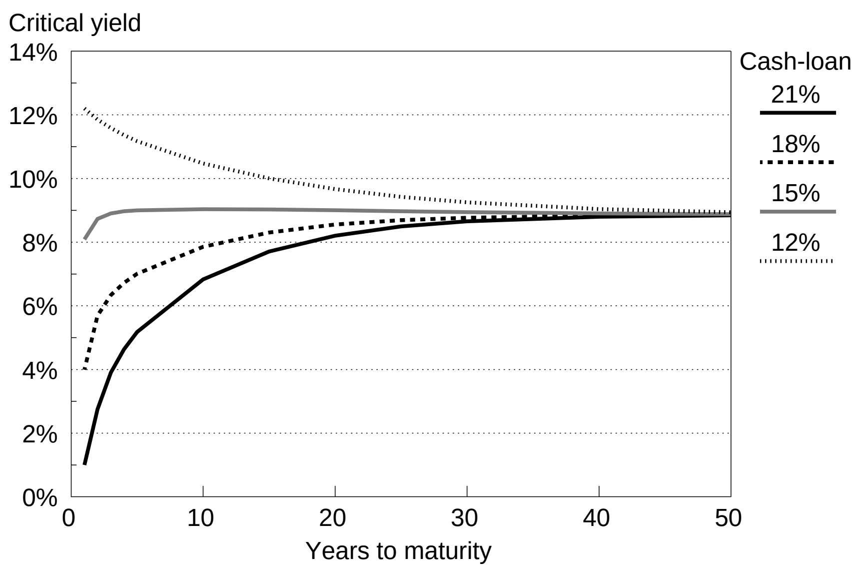

Figure 5.6: Comparison of critical yield and time to maturity for different values of the cash-loan rate.

A cash-loan rate of 12% corresponds to the bond-loan case analysed above. Owners of bond-loans will prepay at lower interest rates as time to maturity increases. This is not necessarily true for cash-loan owners. The cash-loan system increases the share of deductable interest payments relative to total payments, and short term loans are therefore affected most. As seen from the figure, a high cash-loan rate makes prepayment of short term loans very unlikely due to the loss of the tax-subsidy. In extreme cases no positive interest rate exists, which will enable the borrower to prepay. For longer maturities interest payments dominates total payments irrespective of the cash-loan rate, and the impact from the cash-loan system declines converging towards zero with increasing maturity.

The introduction of after-tax calculations complicates the model, and it increases computation times. One might therefore wonder, if after-tax prepayment behaviour could be approximated by a simple pre-tax model. An example of this line of reasoning is found in Dahl(1991), who uses a pre-tax prepayment model with a zero term of notice and a constant cost-rate. Through different choices of the cost rate, he seeks to capture not only the direct costs, but also indirect costs from taxation, term of notice etc.

To investigate this issue, we have calculated implicit cost-rates for different maturities that is the cost-rate in the pre-tax model, which results in the same *initial* critical yield, as the one obtained from the more involved after-tax model.

Table 5.2: *Implicit percentage pre-tax cost-rates corresponding to initial critical yield of a 12% annuity with quarterly payments.*

Time to maturity	Tax-rate 50%, direct cost 0%	Tax-rate 35%, direct cost 0%	Tax-rate 50%, direct cost 2%	Tax 50%, cost 0%, cash-loan rate 18%
1	0.00	0.00	3.90	4.07
2	0.00	0.00	3.77	6.43
3	0.00	0.00	3.65	8.62
4	0.00	0.00	3.54	10.49
5	0.04	0.01	3.46	12.07
10	0.70	0.43	3.62	15.83
15	1.79	1.14	4.44	15.07
20	3.09	1.96	5.57	13.03
25	4.35	2.71	6.75	11.37
30	5.44	3.33	7.81	10.24

The first column of table 5.2 shows the implicit cost-rate for an after-tax model with a direct cost-rate of 0% and a tax-rate of 50%. It is seen that the introduction of taxes has indeed an effect on critical yield similar to a higher cost-rate in the pre-tax model. The implicit cost-rate is dependent on maturity, rising from a value of 0.04% for a 5 year loan to 5.44% for a 30 year loan. The maturity dependency indicates that an approximating pre-tax model, which fits critical yield at the first settlement date, will not be able to fit critical yields at later dates, unless the cost-rate is specified as a decreasing time dependent function as well.

The second column contains implicit cost-rates at a tax rate of 35% and direct costs of 0%. A lower tax rate lowers the implicit cost-rate with the largest effect at longer maturities. The third column shows the effect of increasing the direct cost rate to 2% with a tax rate of 50%. Comparing to the first column it is seen that implicit cost-rates increases by more than the increase in direct costs. This can be explained by the lower volatility of the after-tax mortgage values relative to pre-tax values. Finally the fourth column shows implicit cost rates for the same 12% annuity with a cash-loan rate of 18% and a tax rate of 50%. To approximate the cash-loan one should use average pre-tax cost-rates between 4.07% and 15.83%, but now the cost-rate exhibits a non-monotonous dependency on the remaining time to maturity.

The conclusion from these few examples is that taxation has a specific affect on prepayment behaviour, which is different from the way in which the cost-rate enters the pre-tax model. There is therefore no simple connection between a given after-tax model and the size of the corresponding pre-tax cost-rate. Even assuming such a connection to be found, the pre-tax model should have a time-dependent cost-rate, which probably leads to a kind of model just as complex as the after-tax models. With respect to the term of notice, one should bear in mind that this affects the investor side as well. If borrowers decided to prepay, the value of the bond equals W and not face value.

5.4 The delivery option.

The introduction of differentially taxed investors in a pre-tax equilibrium model opens a Pandoras box of arbitrage possibilities. Section 5.3 showed prepayment to be less likely in an after-tax setting, because by prepaying after a fall in market rates, the borrower looses the tax-advantage inherent in his current higher coupon loan. A similar argument can be made, if market rates rise, but now there is a tax-advantage in switching to a new loan with the coupon rate closer to the market rate of interest.

Switching to a higher coupon loan is possible in the Danish mortgage credit system through the delivery option[54]. At any time borrowers are allowed to buy back the bond equivalent of an earlier issued loan. On delivery of the bonds to the credit institution, the old loans will be cancelled, and the borrower can obtain a new loan. Assuming a pre-tax bond equilibrium, this represents an after-tax gain after a rise in market rates, as discussed in section 4.2.

Conversion to higher coupon rates has been actively marketed by the mortgage credit institutions for several years, but the examples used to promote their services are often obscured by the use of a new callable mortgage as the successor of the old callable loan. In this setting one can picture a no-loose situation, in which the debtor gains from a fall in interest rates by prepaying his 12% loan and refinancing with a 9% loan. When interest rates later rises, the debtor not only gains a tax advantage by exchanging the 9% loan to a 12% loan, he also obtains a loan with a larger prepayment potential and is ready to gain by the next fall in interest rates.

54 The term exchange option would be closer to the Danish term ’ombytning’ normally used, but we feel that the term delivery option provides a better description of the transaction involved.

The delivery option has been analysed in Christensen and Sørensen(1992) by use of numerical examples, but their paper does not model the dynamic debt-management strategies which could be pursued by individual borrowers.

The current section contains a formal analysis of the delivery option. We start with a model for non-callable bonds and show how the exercise value of the delivery option depends on coupon rate, tax-rate, market rate of interest and remaining time to maturity. For MBBs the value of the delivery option depends on the prepayment option as well. By extending the analysis of the previous section, we set up a dynamic debt-management model, in which the borrower determines the optimal strategy with respect to the combined exercise of the prepayment as well as the delivery option. Finally it is shown how the delivery option affects MBB-prices through its influence on borrowers prepayment behaviour.

The first step towards an understanding of the delivery option will be to analyse the exercise value of the delivery option using a simple model with no prepayments. Assume a flat yield curve equal to the continuously compounded rate r. The borrower has issued a non-callable ordinary bond with face value F and a continuously compunded coupon rate c. The borrowers tax-rate equals τ payed continuously. As in the model of the previous section prices are in a pre-tax equilibrium dominated by institutional investors, while the borrower valuates future after-tax payments using the after-tax term structure derived from par bonds with no prepayment or delivery options attached.

Integration of the present value equation leads to the following simple expression for the market price P of a T-year bond with coupon rate c and face value F

(5.4)
$$P = \int_0^T F \cdot c \cdot e^{-rt} dt + F e^{-rT} = F\left[\frac{c}{r} + \left(1 - \frac{c}{r}\right) e^{-rT}\right]$$

Note that par bonds have $c = r$, which is equivalent to $P = F$. The after-tax value of the mortgage, P^{τ}, is equal to

(5.5)
$$P^{\tau} = \int_0^T F(1-\tau)c \cdot e^{-(1-\tau)rt} dt + F e^{-(1-\tau)rT} = F\left[\frac{c}{r} + \left(1 - \frac{c}{r}\right) e^{-(1-\tau)rT}\right]$$

By construction the after-tax value of a par bond equals its pre-tax value. To simplify notation the following analysis will assume $F \equiv 1$.

If the borrower purchased the bond at the market price *P*, he could cancel his loan by delivering the bond to the credit institution. The loan could then be refinanced by issuing par bonds with face value and after-tax value equal to *P*. The exercise value of the delivery option will be denoted $\Pi \equiv P^{\tau} - P$. Inserting (5.4-5) and collecting terms yields the following expression for the exercise value per unit of face value:

(5.6) $$\Pi = \left(1 - \frac{c}{r}\right)(e^{\tau r T} - 1)e^{-rT}$$

The exercise value of the delivery option is a product of three different terms. The first term is related to the difference between the current market rate and the coupon rate. With coupon rates equal to market rates the exercise value will be zero. The second term represents the tax savings on interest payments. If the borrowers tax rate equals zero the exercise value will be zero. The term will be positive for any $\tau > 0$. The value of the delivery option is therefore a pure tax-feature caused by differential taxation. The third term equals the present value of the non-deductable face value. For a perpetuity this term will be zero, and no gain from delivery can be obtained. This simply reflects that all payments on an infinitely long bond are interest payments, irrespective of the coupon rate.

For any $\tau > 0$ and $T > 0$ there is a positive exercise value provided $c < r$ and a negative if $c > r$. In the following we shall assume that $c < r$.

The exercise value depends on four different parameters with partial derivatives given by

$$\frac{\partial \Pi}{\partial c} = -\frac{(e^{\tau r T} - 1)}{r} e^{-rT} < 0$$

$$\frac{\partial \Pi}{\partial \tau} = rT\left(1 - \frac{c}{r}\right) e^{\tau r T} e^{-rT} > 0$$

$$\frac{\partial \Pi}{\partial T} = (r - c)e^{-rT}[1 - (1 - \tau)e^{\tau r T}]$$

$$\frac{\partial \Pi}{\partial r} \equiv \frac{\partial(u \cdot v \cdot w)}{\partial r} = -T \cdot u \cdot v \cdot w \qquad (<0)$$

$$+u \cdot w \cdot \tau T e^{\tau r T} \qquad (>0)$$

(5.7)

$$+v \cdot w . \frac{c}{r^2} \qquad (>0)$$

$$= w\left[uT(1-(1-\tau)e^{\tau r T}) + \frac{c}{r^2} \cdot v \right]$$

The higher the coupon rate on the old loan the lower will be the exercise value, while an increase in the tax-rate yields a higher gain. An increase in time to maturity works in two opposite directions. The gain increases, because more interest payments can be converted to the higher coupon rate on the new loan, but at the same time the gain decreases, as the weight of the non-deductable principal payment relative to interest payments gets lower. The dependence of the exercise value on time to maturity will display a hump increasing up to the value of $T^* = -\ln(1-\tau)/(\tau r)$ and decreasing afterwards. With $\tau = 0.5$ and $r = 0.16$ the maximum gain is attained at $T^* = 8.66$ years.

An increase in the market rate of interest has an even more complex influence on exercise value. Denoting the three positive terms in equation (5.6) as *u*, *v* and *w* respectively it is seen from (5.7) that the gain is lowered due to the increased discounting of the principal payment, while the increased size of tax-deductable interest payments and the increased difference to the old coupon rate tends to increase the gain. In the reduced expression both terms in the bracket are positive for $T < T^*$, which means that the gain will increase at least for maturities below and slightly above T^*, while it may be possible for the gain to decrease with an increase in the market rate of interest for longer term loans.

Figure 5.7 shows the percentage exercise value as a function of time to maturity. As expected we get humped curves due to the diminishing value of the non-deductable principal payment. The exercise value of longer term loans are rather insensitive to the market rate of interest, and for high interest rates a further increase will even lead to a fall in exercise value.

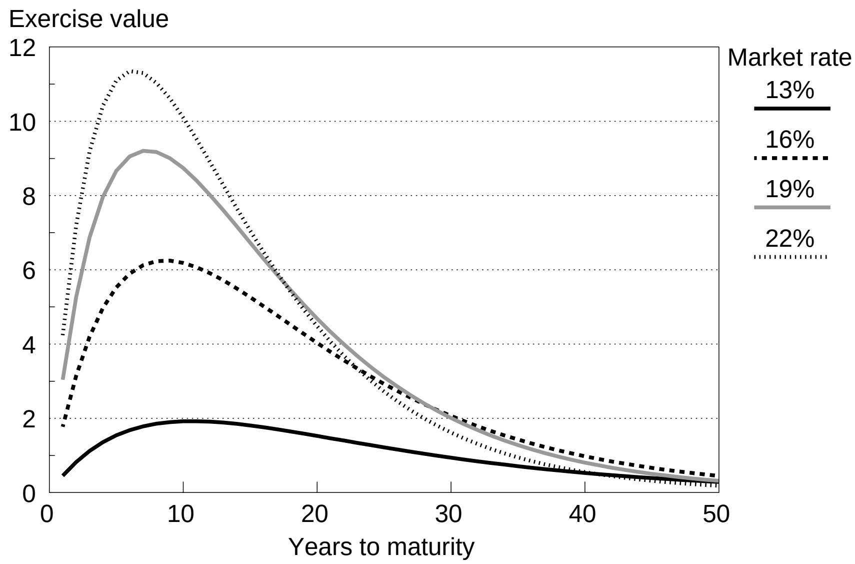

Figure 5.7: Exercise value with continuously compounded ordinary loans. Coupon rate = 12%, tax-rate = 50%.

We now return to the callable annuity with quarterly compounding. Despite the complex nature of the exercise value for the delivery option one can use a simple extension of the prepayment model to study the rational behaviour of a borrower facing the prepayment as well as the delivery possibility.

In the previous sections the mortgage was analysed as a package consisting of a non-callable mortgage and a call option with exercise price equal to the remaining face value. When delivery is taken into account, one simply adds the delivery option, which is an American option, giving the borrower the right to buy back the underlying non-callable mortgage at a price equal to the market price of the callable bond.

At each decision date the borrower has the following options: He could prepay his loan at a cost of W_m, he could continue the loan, the value of which is V_m^+, or finally he could buy back the bond equivalent of the loan, the value of which is equivalent to the market price of the bond, V^+. At delivery the borrower will incur transaction cost assumed to be a constant non-deductable rate ω leaving total delivery costs at $D_m = (1+\omega)V^+$. After-tax present values W_m and V_m^+ are calculated using the par value arbitrage-free method introduced in section 4.3. Assuming the borrower to follow a rational value minimizing strategy, the value of his mortgage at any date-event is given by:

$$(5.8) \qquad V_m = \min\{W_m, V_m^+, D_m^+\}$$

The market price of the callable bond is calculated given the optimal prepayment and delivery behaviour of the borrower along the lines of section 3.3. Note that at delivery the bond is bought at market prices, which means that the only way in which the delivery option will affect market prices is through a change in the borrowers *prepayment* behaviour. This influence will be studied below.

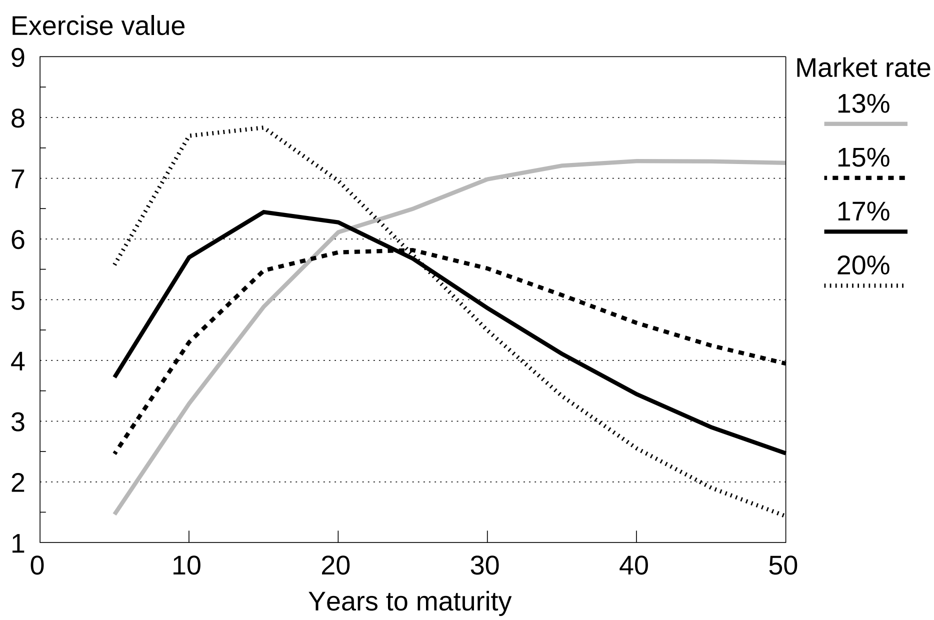

Figure 5.8: Exercise value of the combined prepayment and delivery option for annuity loans with quarterly payments. Coupon rate = 12%, tax-rate = 50%.

Figure 5.8 shows exercise values in a 12% mortgage for different levels of the market rate of interest assuming zero costs of prepayment and delivery. To give an example consider a 15 year mortgage when market rates equals 15%. To deliver the mortgage the borrower should buy callable bonds worth 87.14. This market price is calculated with respect to the prepayment as well as the delivery option. The after-tax present value of the underlying non-callable mortgage is 92.62. The difference of 5.48 represent the after-tax net present value obtained, if he cancels the old 12% loan through delivery and issues a new loan with an effective coupon rate of 15%.

The values given in figure 5.8 represent the combined influence of the delivery as well as the prepayment option. The combined effect is not a simple sum of two individual option values, because the borrower has to exercise both options at the same time. If he exercises the delivery option, he is bound to give up his prepayment option as well and vice versa[55]. As the value of the prepayment option rises with falling interest rates, this explains the high exercise values at a market rate of 13% and 15% compared to non-callable ordinary loans analysed above. The borrower buys back the loan at market values, which are low due to the prepayment risk. The price of a non-callable 15 year annuity at market rates of 15% is 89.19. For a non-callable mortgage the exercise value of the delivery option would thus be 3.43 instead of 5.48.

At higher rates of interest the prepayment option is less valuable, but still there is some differences when compared to the ordinary loans. The maximum gain has moved to longer loans. This could be explained by the earlier repayment of principal in an annuity bond relative to a ordinary bond, which diminishes the maturity effect. Another interesting feature is that the exercise value of a 25 year mortgage is nearly insensitive to the market rate of interest, while the gain decreases with increases in the market rate for loans above 25 years.

Equivalent to the critical yield associated with prepayment we define the *delivery yield* as the pre-tax yield to maturity of the underlying non-callable mortgage at the point where $D_m^+ = V_m$ i.e. where the delivery value of the callable mortgage is equal to its hold-on value.

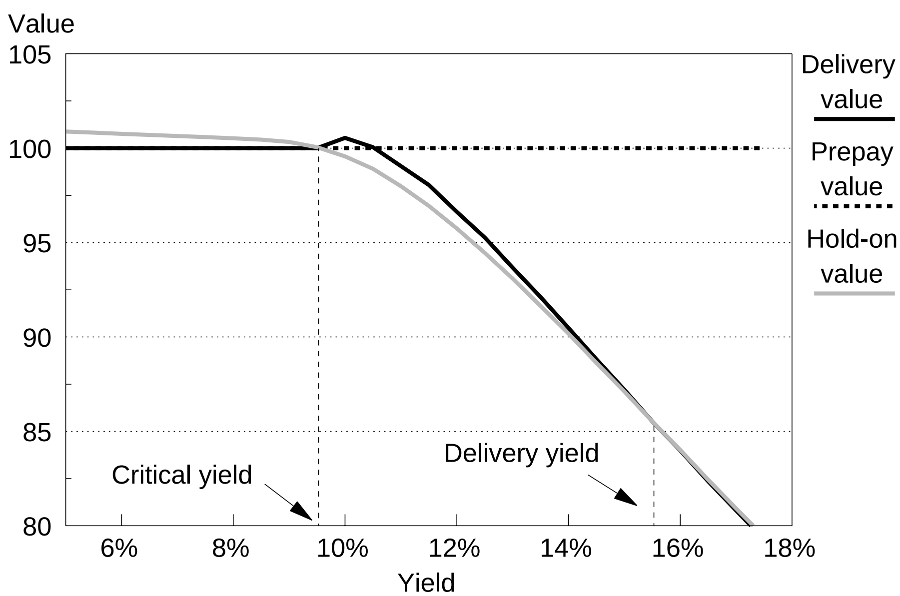

Figure 5.9: Illustration of the combined prepayment and delivery option for a 15 year 12% annuity

55 When the borrower decides to prepay, he gets a loan with a lower coupon rate. One could say that he exercises his delivery option at a loss in order to cash in the gain from the prepayment option.

Figure 5.9 illustrates the decisions facing the borrower having a 15 year 12% annuity mortgage with quarterly payments. The tax rate is 50%. The curve *prepay value* shows the prepayment value of the mortgage, while the curve *delivery value* shows the value, if the borrower decides to exercise his delivery option. Assuming zero costs the two are equal to remaining principal of the mortgage and the market price of the callable bond respectively. The curve *hold-on value* is equal to the after-tax value of the mortgage, given that the borrower continues his mortgage to the next decision date, and assuming that he follows an optimal value minimizing strategy afterwards. At interest rates below critical yield, the borrower will prepay his mortgage as discussed above.

When interest rates rise, the after-tax value of the remaining mortgage payments rises relative to their pre-tax value, because the coupon rate becomes lower than the market rate of interest. At the point labeled ’delivery yield’ the hold-on value of the mortgage is equal to the delivery value, and the borrower will purchase the bond equivalent at market prices, and cancel his old loan through the delivery option.

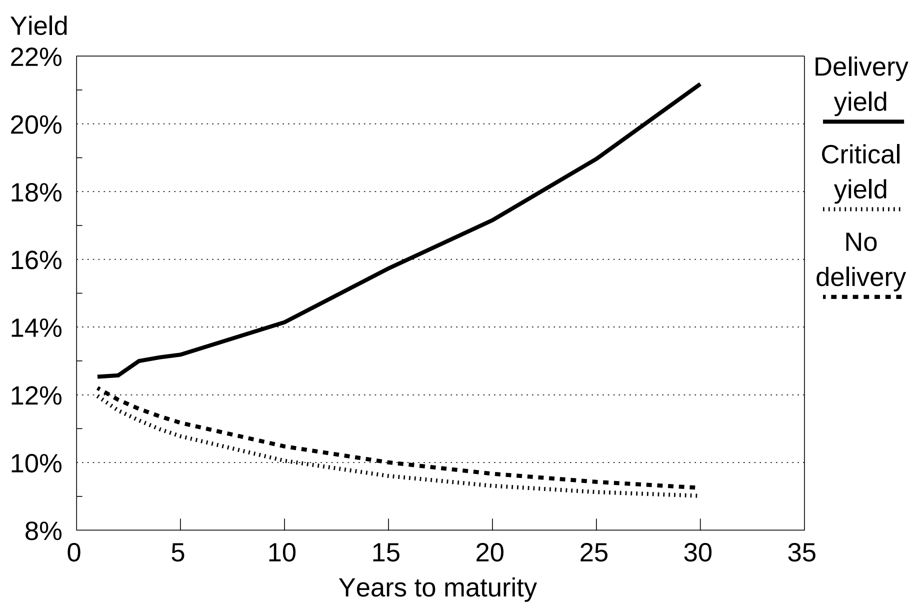

Figure 5.10: Delivery yield and critical yield for different maturities.

Figure 5.10 shows percentage delivery yield for different maturities assuming zero transaction costs. For comparison we have plotted the critical yield calculated with and without the delivery option. The exact numbers are shown in the following table.

The delivery yield is shown in column (a) of the table. For a very short term mortgage the rational borrower should use his delivery option, as soon as the market yield rises above the effective coupon rate of the mortgage. For longer term mortgages the bor-

Table 5.3: *Delivery yield and critical yield with and without the delivery option for different maturities.*

Time to maturity	Delivery yield (a)	Critical yield with delivery (b)	Critical yield with no delivery (c)	Bond value at delivery (d)	After-tax value to maturity (e)	Gain from delivery in % of after-tax value (f)
1	12.53	11.96	12.20	99.89	100.00	0.12
2	12.57	11.54	11.85	99.60	99.99	0.39
3	13.00	11.24	11.58	98.96	99.70	0.74
4	13.10	10.98	11.36	98.38	99.50	1.12
5	13.18	10.78	11.17	97.75	99.29	1.55
10	14.14	10.05	10.47	92.72	96.58	3.99
15	15.73	9.60	10.00	84.89	90.60	6.30
20	17.15	9.31	9.67	77.41	83.73	7.55
25	18.97	9.13	9.42	69.72	75.43	7.57
30	21.18	9.01	9.25	62.30	69.67	10.58
40	26.11	8.89	9.04	50.65	55.34	8.48
50	30.82	8.84	8.94	43.29	46.77	7.43

rower takes the time value of the delivery option into account. The delivery option embedded in a 10 year mortgage should be exercised at 14.14%, while the delivery option in a 20 year mortgage should be exercised at 17.15%. For comparison column (d) shows the value of the callable bond at delivery, while column (e) shows the corresponding after-tax value to maturity of the mortgage. Column (f) shows the exercise value as a percentage of after-tax value.

Delivery yield for longer term mortgages may seem rather large. This could perhaps be explained by the complex connection between time to maturity and the market rate of interest discussed above. The rational borrower values the exercise value now against delivery at the next decision date. As exercise value increases, when time to maturity decreases, the loss from waiting is small.

Perhaps the most interesting question will be how and if the delivery option affects prices of bonds. At delivery the bonds will be bought at the prevailing market price, and in that respect the bond owners are not affected. But there is a small effect due to the way in which the delivery option affects prepayment behaviour. The possibility of delivery reduces the hold-on value of the mortgage and thereby reduces the critical yield. To put it differently, the borrower is less inclined to prepay his mortgage due to the fact that the delivery option makes it less costly to keep the mortgage in case of a later rise in interest rates. As shown in figure 5.10 and table 5.3 this effect is rather small however, especially for longer term mortgages. But according to the current analysis the delivery option should increase prices of callable mortgage backed bonds.

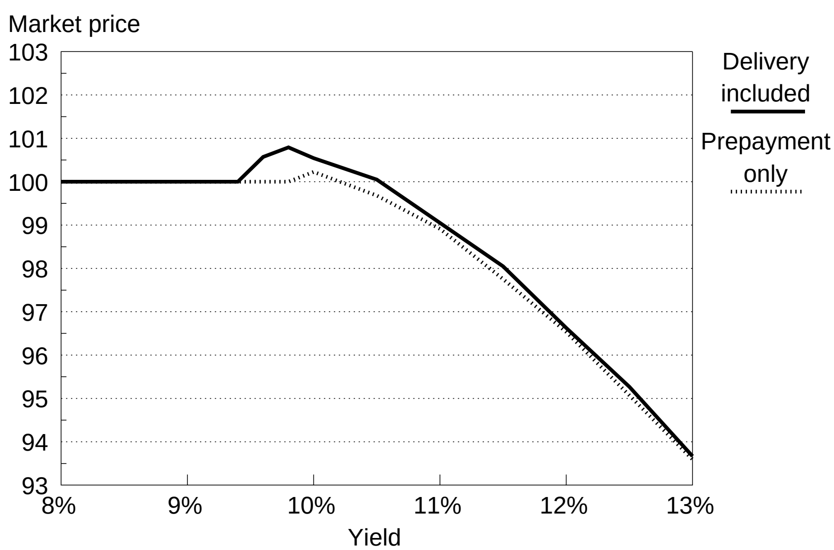

Figure 5.11: Comparison of MBB-prices with and without the delivery option.

Figure 5.11 compares prices with and without the delivery option for the 12% 15 year MBB studied above. As expected, prices calculated with respect to both options are a little above prices with the prepayment option alone. The differences are barely visible amounting to at most 37 basispoints except for the area between the critical yield with and without delivery where the differences increases to 79 basispoints. Introducing positive delivery costs should tend to close the gap even further. The conclusion is therefore that the delivery option increases MBB-prices, but the effect is small and can be safely ignored.

This is a partial view of mortgage finance. A more practical inclined reader could probably reach the opposite conclusion. It could be said that if the borrower prepays the 12% mortgage issuing a 9% mortgage instead, he gets a new delivery option, which due to the lower coupon rate of the 9% loan is more valuable than the old one. If prepayment were valuable without taking the delivery option into account, then it is even more attractive afterwards, because the tax-advantage lost by prepaying can be regained at a subsequent rise in interest rates. This should lead to a rise in critical yield and a fall in MBB-prices.

The difference between the two conclusions hinges on the assumptions made regarding the availability of the delivery option. In the analysis above we tacitly assumed the delivery option embedded in the old loan to be a scarce resource and valued it against an after-tax lattice of bonds with no delivery options attached. But the delivery option is not a traded option which demands a premium. Instead it is a part of the tax-system, working in a complex way, but otherwise similar to a direct tax-subsidy to the borrower. It can therefore be argued that the after-tax lattice and the after-tax valuations should be modified to reflect the fact that all loans could be delivered after a rise in interest rates. How and if this modification should be done must await future research.

5.5 A critique of the American option model

The American option model has provided valuable insight into optimal prepayment behaviour of the individual borrower and as shown in section 5.3 and 5.4 it can be extended in several directions. Some authors have used the American option approach as a full pricing model for Danish MBBs. Dahl(1991) uses a pre-tax BDT-model similar to the one discussed above. Christensen(1985) uses a one factor continuous time Cox, Ingersol and Ross(1985) model to price the MBB. He explicitly ignores transaction costs, but includes taxes, which as shown above give rise to similar results. Mouritsen and Møller(1987) let yield to maturity follow a binomial process and use this to price MBBs in an after-tax framework. And finally we have used the same approach in the analysis of the delivery option above.

A problem common to these models is the erratic price behaviour stemming from the binary prepayment function (5.3). At a decision date the borrower prepays, $\lambda = 1$, if the prepay value is less than the hold-on value of the mortgage, $W_m < V_m^+$, and he continues his loan, $\lambda = 0$ if $W_m \geq V_m^+$. The resulting value of his mortgage can be written as $V_m = \lambda W_m + (1 - \lambda)V_m^+$, which is a continuous function of W_m and V_m^+, since the jumps in λ happens, where the two values coincide. Seen from the investors point of view there will however be a discontinuity at the critical yield, because the hold-on value of the bond V^+ will typically be greater than its prepayment value W, at the point where the borrower chooses to prepay. The only exception to this discontinuity is the pre-tax model with a zero cost-rate, in which the mortgage value coincides with the bond value V at all future date-events.

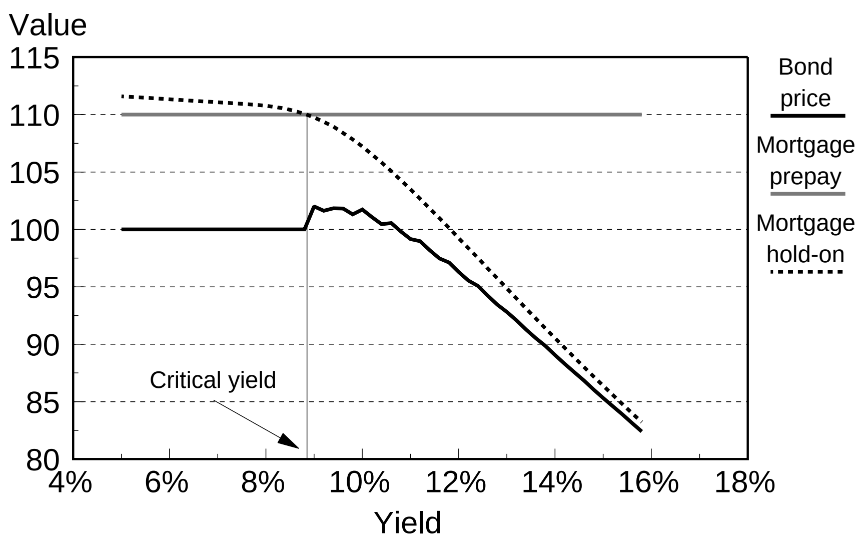

Figure 5.12: Plot of price behaviour in response to discontinuous prepayment functions

The problem is illustrated in figure 5.12 for the standard 20 year 12% annuity with a tax-rate of zero and a cost-rate of 10%. The term structure model has been computed with 32 time steps per year. At the critical yield of 8.9% all borrowers prepay their loans. This leads to a sharp 2 per cent drop in the price reflecting the loss to investors of an extra settlement period at the higher coupon rate. Even at yields above the critical yield prices display a jagged pattern reflecting discontinuities at later settlement dates.

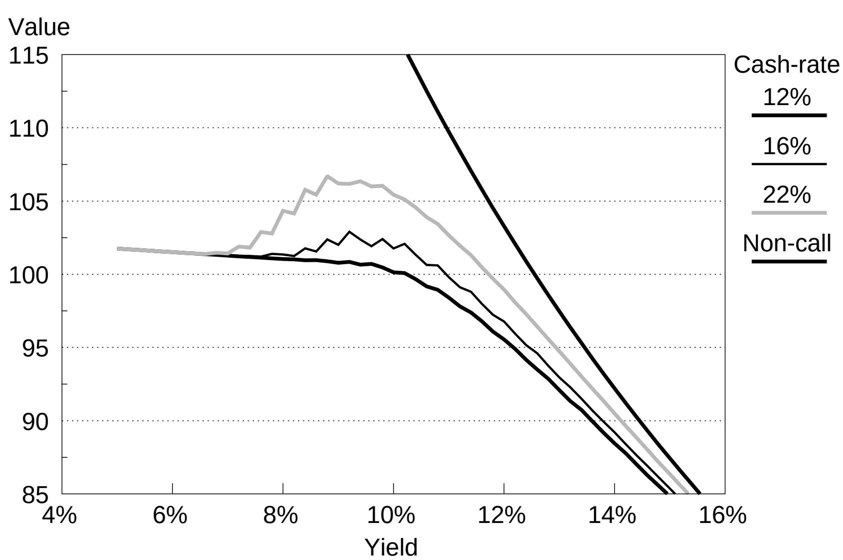

Figure 5.13: Plot of price behaviour in response to discontinuous prepayment functions, for different levels of the cash-loan rate

Figure 5.13 shows calculated bond prices for various cash-loan rates assuming zero costs and a tax rate of 50%. Even though computations are done with 3 months remaining to the next decision point, the figure reflects the upcoming discontinuities.

Increasing the number of time steps could smoothen away most discontinuities at the expense of longer computation times. The basic discontinuity at the decision dates would still exist however, and prices would still be highly variable close to the critical yield. Problems are aggravated, if the model is used to calculate option adjusted duration or convexity, as a small change in yield can result in large changes in size or even sign of these measures.

The discontinuity problem has been documented in Christensen(1985) as well as in Mouritsen and Møller (1987), who suggest that the problem could be solved by averaging over several groups of borrowers characterized by differences in transaction costs. This procedure helps to reduce the discontinuities, because we now have several critical rates, and at each rate only a minor share of all borrowers prepay.

In section 6 and 7 we present models, for which the discontinuity problem has been removed through the use of a continuous prepayment functions.

5.6 Conclusion

In this section the callable mortgage backed bond has been modelled as a non-callable bond with an embedded American call option, giving the borrower a right to prepay his loan at any date before maturity. Optimal prepayment behaviour has primarily been studied in terms of the critical yield at time zero that is the market rate of interest, at which the borrower decides to prepay his loan. In a pre-tax setting we have studied the influence of time to maturity, transaction costs, the coupon rate and the term of notice on critical yield, and in general it was shown that the mortgage will be prepayed at market rates well below the effective coupon rate.

The arbitrage-free after-tax model developed in section 4.3 was used to analyse the impact of tax considerations on prepayment behaviour. As interest payments are fully tax deductable, prepayment was shown to be less attractive on an after-tax basis. Especially the favorable tax status of cash-loans becomes an effective barrier against prepayment.

The computation of implicit pre-tax cost equivalents for a range of after-tax models revealed that taxes enter the prepayment decision very differently from the way in which transaction costs enter the pre-tax model. This makes the use of cost-adjusted pre-tax models impractical as a substitute for an after-tax valuation of callable mortgage backed bonds - especially for cash-loans.

The delivery option enables the mortgage owner to cancel his loan at market prices in exchange of a new loan at a higher coupon rate. The value of the delivery option was an example of a dynamic tax-arbitrage possibility made possible by the coexistence of differentially taxed investors in the same market. Our study of the exercise value of the delivery option showed that its value depended negatively on the coupon rate of the old loan and it increased with the borrowers tax-rate. As a perhaps counter-intuitive result the value of the delivery option peaked for loans around 10 years to maturity. Likewise it was shown that a higher market rate might diminish the gain from delivery.

The analysis of the delivery option for MBBs is complicated by the prepayment option. We have developed a formal model in which the borrower on an after-tax basis decides on the combined exercise of his delivery as well as his prepayment option. Optimal use of the delivery option was shown to affect prepayment behaviour, although the influence on bond prices was judged to be minor.

It was finally shown that the binary prepayment function used in the American option model gives rise to discontinuities in price behaviour around the decision point. As discussed in the next two chapters, this discontinuity can be removed by the introduction of a continuous prepayment function.

6 A Prepayment Model with Heterogeneous Borrowers.

In this chapter we propose a model for the pricing of callable mortgage backed bonds that allows for heterogeneous borrowers and is closely connected to current advisory practice in the Danish mortgage institutions. We assume that while the current gain from prepayment is readily available from the credit institutions, borrowers differ in their assessment of the future value of the prepayment option. To account for actual prepayment behaviour the future value of the prepayment option is assumed to follow an exogenously specified continuous distribution. This implies a continuous prepayment function and a correspondingly smooth behaviour of prices and risk measures.

In a previous paper, Jakobsen(1992), the same prepayment function has been discussed within the context of a binomial process for the yield to maturity of the underlying non-callable bond. The current model is based on an arbitrage-free BDT-model, which among other things allows us to study how the shape of the initial term structure affects bond prices.

The chapter starts by a critical discussion of the "optimal" exercise concept used in chapter 5. Section 6.2 formally states the model, while price behaviour and derived risk measures are the subject for section 6.3 and 6.4 respectively. In 6.5 a small empirical analysis is conducted, which points to the importance of data on the composition and tax-status of borrowers in the individual bond issues. A brief summary of the results can be found in section 6.6.

6.1 Optimal versus actual prepayment behaviour.

The previous chapter used the theory of American options to determine an optimal value minimizing strategy to be used by the rational mortgage owner. The approach, common to several other works in this area, is in close accordance with economic theory, but several objections could be levered against it as a description of actual prepayment behaviour.

We will start by restating the notation of the previous chapter with small adjustments. B_m is defined as the after-tax value of the underlying non-callable mortgage that is the remaining after-tax payments of the mortgage assuming no prepayments, discounted by the after-tax term structure. W_m denotes the prepayment value, equal to current face value plus prepayment costs and costs associated with the term of notice. The hold-on value of the mortgage is denoted V_m^+. V_m^+ was calculated as the value of future after-tax payments assuming an optimal strategy to be followed at all future decision dates. Given optimal prepayment behaviour the borrower prepays, if $V_m^+ > W_m$, and continues otherwise. This leaves the value of the mortgage equal to $V_m = \min\{V_m^+, W_m\}$. The critical yield r^* was defined as the pre-tax yield to maturity of the underlying mortgage, where $V_m^+ = W_m$.

The exercise value of the prepayment option is defined as $C_m^* \equiv B_m - W_m$. This represents the present value of future after-tax savings, if the borrower prepays his old loan and refinances with a new loan at the market rate of interest. The gain from prepayment could become negative, if the borrower strikes the prepayment option early. We finally define the *gain from prepayment*, $g_m = (B_m - W_m)/B_m$, i.e. the ratio between the exercise value and the value of the underlying non-callable mortgage. The percentage gain from prepayment is perhaps the statistic most commonly used by the advisory services of the mortgage credit institutions.

Note that when the borrower prepays his loan, we have $V_m^+ = W_m$, implying that the exercise value of the prepayment option is equal to the value of the future prepayment possibilities, $B_m - V_m^+$. The gain from prepaying at the critical yield is therefore equal to the borrowers estimate of the future value of the prepayment option as a percentage of a similar non-callable mortgage. We shall denote the last number as the *future gain from prepaying*, $g_m^+ \equiv (B_m - V_m^+)/B_m$.

The first objection to the American option model is the amount of information needed to calculate the future value of the callable mortgage V_m^+. The borrower should be aware of the precise terms of his mortgage, be acquainted with the zero coupon term structure of interest rates, be able to specify the structure and volatilities of future short term interest rates and finally perform the backwards pricing procedure.

Assume for a moment that all these prerequisites are at hand, except for some uncertainty regarding the future interest rate volatility. Dahl(1991) obtains possible volatility estimates of 10%, 14% and 16% from different sources and chooses 15% in his work on MBBs. It is probably fair to argue that the average mortgage owner faces at least the same level of uncertainty. To see how uncertainty on volatility affects prepayment behaviour, we have calculated critical yields for a 12% annuity mortgage[56]. The figures are given in table 6.1.

Table 6.1: *Critical yield and percentage gain from prepaying for different values of volatility and time to maturity.*

Time to maturity	Critical yield			Percentage gain at prepayment		
	Volatility			Volatility		
	12% (a)	15% (b)	18% (c)	12% (d)	15% (e)	18% (f)
15	10.39	9.96	9.55	7.69	9.19	10.59
20	10.18	9.70	9.26	10.27	12.32	14.20
25	10.01	9.51	9.06	12.68	15.18	17.43
30	9.90	9.38	8.92	14.74	17.58	20.11
40	9.75	9.23	8.76	17.74	21.04	23.94
50	9.68	9.16	8.70	19.53	23.09	26.21

The uncertainty regarding future volatility has a rather large impact on critical yield, especially for the longer term loans. Column (a)-(c) shows that a shift in perceived volatility from 12% to 18% for a 30 year mortgage lowers critical yield from 9.90% to 8.92%. Column (d)-(f) shows the percentage gain from prepaying at the critical yield. For the 30 year loan it is seen that when the borrower perceives a future volatility of 12%, he prepays at a gain of 14.74%, while he prepays at a 20.11% gain, if future volatility is estimated at 18%.

Uncertainty regarding the interest rate process, the initial term structure, future tax schemes etc., could all contribute to similar ranges of possible values for V_m^+. But the largest problem probably lies in the hidden assumptions of the entire concept of an option adjusted mortgage value.

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[56] Calculations are done with a flat initial term structure and 8 steps per year. The borrower is assumed to be a corporation with a tax-rate of 38%, a cost-rate of 0.5% and a 3 months term of notice. As usual results are for time to maturities up to 50 years to give an impression of asymptotic behaviour.

Consider the mortgage owner in the example above, having a 20 year annuity mortgage. Volatility equals 15% and the market rate is 11%. At these values $B_m = 107.61$, $W_m = 100.41$ and $V_m^+ = 98.46$. The gain from prepaying is 6.69%, while the future gain, g_m^+ is equal to 8.51%. As $g_m^+ > g_m$ the rational value minimizing borrower should continue his loan. But how does he cash in the future gain of 8.51% ?

The argument from standard option theory would say that if future value is larger than exercise value, then the option should be sold at the market instead of being exercised. As an alternative one could set up a delta-hedge strategy.

The prepayment option is embedded in a mortgage. Reselling the prepayment option would normally imply that the borrower should sell the real estate being put as collateral. Transaction costs alone prohibit this.

A delta-hedge strategy could be designed according to the refinancing alternative of the borrower. The borrower has a long position in a call-option and a short position in the underlying non-callable bond. Consider first a borrower, who wants to prepay in order to refinance with a new non-callable loan at the market rate of interest. As an alternative to prepaying above the critical yield he could add a short position in a non-callable bond and a long position in the money market. The new combined position should have an after-tax value equal to prepayment value and a delta equal to the delta of a new non-callable loan. The arbitrage-free pricing principle ensures that the combined position could be established with a net gain of $W_m - V_m^+$. To keep the position in line with the alternative the borrower should adjust his position, when interest rates changes. If interest rates fall below critical yield, the borrower prepays the mortgage and takes a new non-callable loan.

If the alternative is to prepay and refinance with a short term loan the borrower should reverse the procedure adding a long position in a non-callable bond and short position in the money market. The combined portfolio could be chosen with a value equal to prepayment value and a delta of zero. The arbitrage-free principle would ensure a net gain of $W_m - V_m^+$. The position should be continuously adjusted as long as interest rates are above the critical yield.

Consider finally a borrower, who wants to prepay and refinance with a new callable mortgage with a market value of $\overline{V} = W_m$. The borrower should first perform an after-tax valuation of the new mortgage resulting in the value of $\overline{V}_m$. If $\overline{V}_m = \overline{V}$ then the situation is similar to the case with a non-callable loan as the alternative except for some adjustments due to the lower delta of the new callable bond. The borrower could still obtain a net gain of $W_m - V_m^+$ by delta-hedging instead of prepaying. If $\overline{V}_m < \overline{V}$ then a new situation arises, because the issuance of the new loan represents a separate arbitrage possibility. If $\overline{V} - V_m > W_m - V_m^+$ the gain from the new loan outweighs the loss from early prepayment, and it could be argued that the borrower should prepay his old loan at a loss. This implicitly assumes however that the arbitrage possibility is temporary, and that the borrower faces restrictions, which unables him to gain without giving up the old loan.

In principle the borrower can at any time lock in the future gain from prepayment through delta-hedging. As in standard option theory the delta-hedge argument provides a preference free valuation procedure. The opposite is also true however. If the borrower cannot perform the delta-hedge strategy, then values obtained from an option pricing procedure has little credibility.

There are several objections to the delta-hedge procedure. All three hedge strategies increase gross loan volume for the individual borrower. Except perhaps for large corporations the typical borrower would incur transaction costs as well as restrictions on credit. Problems are aggravated due to the uncertainty regarding individual parameters and thereby the profitability of the strategy, cf. above. And neither the borrower nor his advisors have access to the relevant computations.

Table 6.2 shows the percentage gain from prepaying (d) and the future gain from prepaying (e) at different levels of the market rate of interest. The difference (f) represents the gain from delta-hedging. For interest rates at or below 11% the difference between prepaying and delta-hedging is so small that even small transaction costs would make delta-hedging more costly than prepaying. This invalidates the basic argument behind the determination of the critical yield, namely that all borrowers would wait to prepay if $W_m > V_m^+$, because anyone who considered prepaying, would prefer the lower value of the delta-hedge strategy. Instead we are left with a range of possible prepayment points, depending on the borrowers subjective assessment of the required gain. At interest rates above 11% delta-hedge could be an alternative to prepaying, but most borrowers would probably not consider any of the alternatives, due to the small size of the option value.

To summarize we have argued that the American option approach to prepayment behaviour rests on rather weak foundations. It assumes a knowledge and a computa-

Table 6.2: *Comparison of percentage gain from prepayment to future gain from prepayment for different levels of non-callable yield to maturity.*

Yield to maturity	After-tax value of non-callable mortgage (a)	Prepay value (b)	Hold-on value (c)	Gain from pre-paying % (d)	Future gain from prepaying % (e)	Difference in gain (f)
8%	125.27	100.84	101.47	19.50	19.00	-0.50
9%	118.90	100.70	101.09	15.31	14.98	-0.33
10%	113.03	100.56	100.20	11.04	11.35	0.31
11%	107.61	100.41	98.46	6.69	8.51	1.82
12%	102.60	100.27	95.91	2.27	6.52	4.25
13%	97.96	100.13	93.04	-2.22	5.02	7.25
14%	93.65	100.00	90.04	-6.77	3.85	10.63

tional ability unavailable to most mortgage owners, and the choice between prepayment and a delta-hedge strategy is not a convincing one, due to the transaction costs involved in the latter.

6.2 A stochastic model of prepayment behaviour

The American option models assumption of homogeneous borrowers implied a binary prepayment function with either zero or full prepayment. This resulted in a discontinuous price behaviour. As shown in the previous section the AO-model furthermore assumed borrowers with full information and no transaction costs or credit restrictions. In the real world we would expect heterogeneity among borrowers as well as uncertainty concerning the future gain from prepaying.

The current section introduces a model with a continuous prepayment function. The model is based on a stochastic ’micro-economic’ description of individual borrowers behavior. Contrary to the AO-model borrowers are no longer assumed to behave entirely rational. The AO-model will still be used however, to argue for the inclusion of explanatory variables as well as the choice of shape for the prepayment function.

Several authors have proposed models for US mortgage backed securities, in which the borrowers are assumed to prepay stochastically as opposed to following a rational exercise behaviour. Green and Shoven(1986), Schwartz and Tourus(1989) and McConnel and Singh(1991) use the proportional hazard model by Cox(1973) assuming that each borrower has a probability of prepaying depending on remaining time to maturity and conditioned on the current state of the economy. By the inclusion of explanatory variables these authors are able to model how mortgage age, interest rates, seasonality etc. affect the prepayment rate. Richard and Roll(1991) use a more direct regression model to the same effect. The US models are reviewed in chapter 7.

Gain from prepayment is used as the primary determinant of prepayments. As mentioned earlier this concept is readily available to the mortgage owner. Each mortgage pool is assumed to consist of a large number of individual mortgages. At each decision point mortgage owners observe the gain from prepayment g_m. Conditioned on this gain a fraction $\lambda(g_m)$ of all borrowers decides to prepay. Note that $\lambda(g_m)$ is equal to the prepayment function introduced in section 3.3. The exact shape of the prepayment function is of course an empirical matter, but we would expect the prepayment function is monotonously increasing starting at zero for very low values of g_m and converging toward one for large values of g_m. Continuous distribution functions are an obvious class of functions satisfying these properties. The use of a distribution function will furthermore allow an interesting 'micro-economic' heterogeneity-interpretation of the model as shown below.

We assume, with no claim of originality that the prepayment function can be described by a normal distribution function, $N(\cdot\,; g^*, \sigma)$, with mean g^* and a standard deviation of σ. As shown in the next chapter this leads to an empirically tractable model, which can be tested on available prepayment data.

As one obvious interpretation of the model assume each individual borrower i to have a *required gain*, g_{im}, which induces prepayment. If $g_{im} \leq g_m$ the borrower prepays, otherwise he waits for a further fall in interest rates. Assuming the required gain to be normally distributed across the population of borrowers with mean g^* and standard deviation σ implies that the fraction of borrowers who prepay at a gain of g_m will equal $N(g_m; g^*, \sigma)$. In this model heterogeneity with respect to required gain leads to partial prepayment at each decision date, as opposed to the binary prepayment function of the American option model.

A second interpretation could be that all borrowers are homogenous, but their assessment of future gain are subject to uncertainty due to all the factors listed in the previous section. If the individual valuations at a given point in time are normally distributed, a similar prepayment function would be appropriate.

The rest of this chapter will use the heterogeneity interpretation with a normally distributed required gain. The normal distribution function could of course be substituted by almost any continuous distribution function.

The price of the MBB can be found by the general approach described in section 3.3. At each date-event one computes the gain from prepayment $g_m \equiv (B_m - W_m)/B_m$. The values B_m and W_m at each date-event can be efficiently updated as part of the backwards pricing procedure. For after-tax calculations we use the arbitrage-free valuation approach described in section 4.3. The gain from prepayment is inserted into the assumed distribution function resulting in a prepayment rate $\lambda(g_m)$. The value of the bond at a single date-event is given by

(6.1) $$V = \lambda(g_m) \cdot W + (1 - \lambda(g_m))V^+ \quad ,$$

with V^+ computed by the backward equation (3.20).

The prepayment rate is now a continuous function of the market rate of interest and other parameters of the model. The discontinuities present in the American option model are therefore absent in the present model even though W and V^+ will in general be different at the decision date.

As a one-plot-fits-all illustration of the model we have plotted the price of a 12% coupon 20 year MBB as a function of the market rate of interest[57]. The price of a similar non-callable bond, B, and the prepayment value, W, has been plotted for reference. To elaborate further the lower part of the plot shows the initial after-tax gain, g_m, and the resulting prepayment rate.

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[57] Borrowers are assumed to have a taxrate of 50% with non-deductable costs of 0.5% and a 3 months term of notice. The mean value of the required gain is a constant g^* equal to 15 percent with σ equal to 3 percent. Interest rate volatility is set at 10% and we use 8 steps per year.

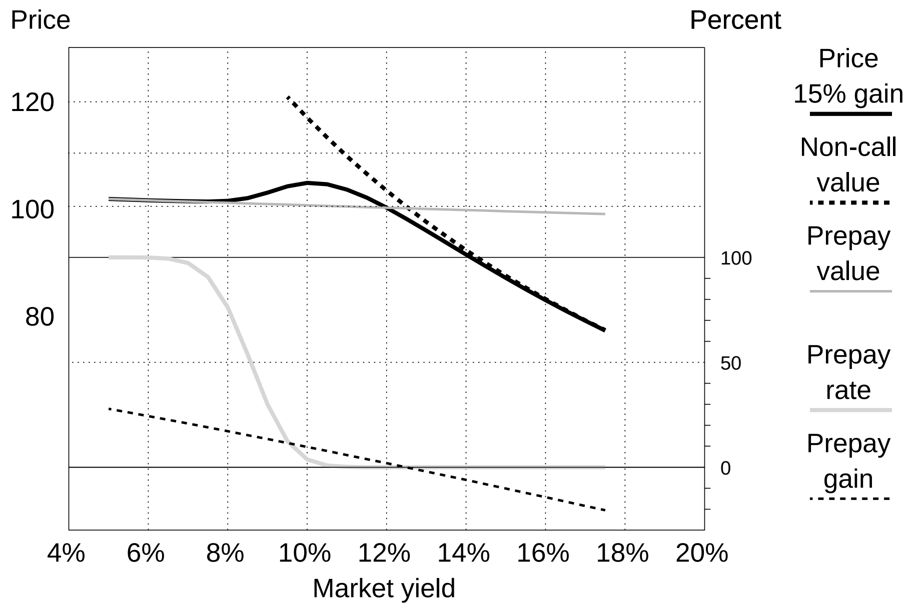

Figure 6.1: An illustration of the prepayment model.

As expected prices vary continuously with the market rate of interest. At market rates above 14% prepayment is still a distant possiblity and the MBB price are close to the price of a similar non-callable bond. As interest rates fall the gap between the MBB and the non-callable bond widens due to the increased probability of prepayment in the near future. Between 11 and 10% some borrowers start to prepay their loans and at a market yield of 10% the MBB reaches its maximum value of 104.74. Between 10 and 8% the MBB exhibits negative duration because the gain in value of the underlying non-callable bond is offset by the increased prepayment risk. At interest rates below 8% nearly all borrowers prepay their loans with 3 months term of notice, and the value of the MBB equals the value of a three months 12% bond.

Large differences in prepayment behaviour between borrowers with different tax schemes or different loan sizes are unlikely to be described by a normal distribution. In these cases one can group borrowers according to basic parameters and apply the valuation approach for each group individually. The resulting value of the bond can be found as a weighted average. For an example se section 6.5 below.

Note that the American option model could be seen as a limiting case of the current model with a standard deviation of zero and a mean required gain equal to the gain obtained at the critical yield. This analogy will be used repeatedly in section 6.3, when we discuss how a rational borrower would adjust his required gain to changes in the economic environment. But although rational prepayment behaviour presents a starting point, it is the actual behaviour which is of interest to the investor. In the remaining part of this section we discuss a number of reasons, why borrowers prepay at non-optimal gains, and how these effects could be presented in the present model.

The most important reason was given in the previous section noting that the entire concept of a single optimal gain was invalidated by transaction costs, computational problems and uncertainty of future volatility and term structure estimates. Incorporating these factors into an American option model would leave us with a large range of possible prepayment point, neither of which could be deemed the most correct to use. In the required gain model this uncertainty is simply modelled through an appropriate choice of σ.

The Danish mortgage market is highly regulated and restrictions on credit could make borrowers focus more on short term net cash flow as opposed to the long term present value gain. Figure 6.2 illustrates two important examples. A change of legislation in 1986 forced all new mortgages to be issued as so-called mix-loans with 40% being issued as serial bonds and 60% as annuity bonds. Before 1986 all mortgages were issued as annuities. The curve labelled ’Annuity 12%’ shows annual after-tax cash flow of a 30 year 12% annuity bond with a face value of 500.000 DKK. At market rates of 9% this bond could be prepayed with a present value after-tax gain of 8.53% net of transaction costs. After prepayment the borrower would normally take a new 30 year 9% mix-loan, labelled ’Mix-loan 9%’. Despite the gain in present value the borrower faces increased costs on his mortgage for the first few years. Borrowers with mix-loans issued after 1986, like the one labelled ’Mix-loan 12%’ on the figure, would face a uniform reduction in payments. To incorporate short-sighted liquidity concerns into the present model we could assume a larger required gain for borrowers with annuity mortgages relative to the mix-loan mortgage owners.

Borrowers, who prepayed under the Danish mortgage system up to 1992, were legally bound to issue a new loan with time to maturity less than or equal to the old loan. Switching to longer term loans was only allowed in connection with a sale of the property. In May 1992 the credit restrictions were eased, and provided sufficient collateral homeowners could for the first time prepay and take a *longer* term loan. This has induced a large amount of borrowers to prepay shorter term loans like the 10% 15 year loan with a cash-loan rate of 16% shown in the figure. As seen the borrower gains

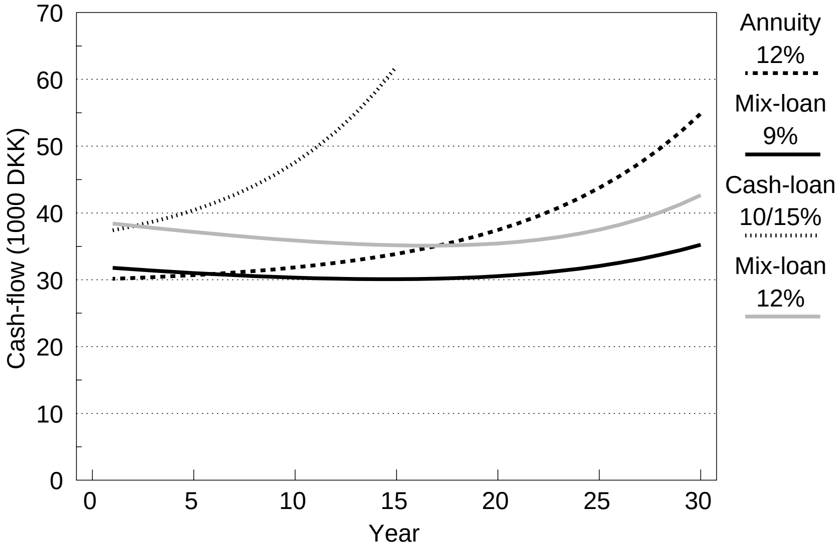

Figure 6.2: Annual after-tax cash flow from different mortgages. Tax-rate 50%, face value 500.000 DKK. The market rate of interest 9%.

a reduction in after-tax payments for the next 15 years, but in present value terms he actually looses 6.60%[58]. This change of legislation could be incorporated as an increased probability of prepaying at a negative gain.

Turnover of houses have similar effects. As opposed to the typical American mortgage system the Danish mortgages are contractually assumable to a new owner and no forced prepayment takes place at sale. Most houses are advertised on first years payments after tax however, which gives some incentives to prepay short term loans perhaps at a net present value loss and refinance with longer term loans. For a discussion of the connection between turnover rates and the rate of prepayments see Mouritsen and Møller(1989).

A kind of money illusion may exist, which induces corporate owners of large mortgages to prepay at lower gains than private houseowners, simply because a 4% gain on a 100 million DKK loan looks more impressive than a 10% gain on a 200.000 DKK mortgage. To account for these effects we should use a lower gain for larger mortgages. Likewise one would not expect a very small mortgage of say 25.000 DKK to be prepayed irrespective of the possible gain.

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[58] Sources at one of the credit institutions estimates that more than 50.000 mortgage owners have made inquiries about longer term loans in the first few month after the bill was passed. The final figures have not been collected but a conservative estimate would be that they represents outstanding mortgages of 5-10 billion DKK. A neighbourhood effect could make this type of prepayments rather epidemic as large housing areas build in the 70'ties have been financed with the same kind of mortgages.

In recent years there has been an active marketing of prepayment services from banks and mortgage institutions. The latest offerings are loan watch services, in which the borrower contractually allows the credit institution to prepay as soon as certain conditions are met. The conditions are normally put in terms of a required gain, which could be fed directly into the current model. Regrettably no public information exists on the number and gain distribution of these contracts, but it would be a fair bet to assume that a high proportion of watched loans could be modelled as a reduction in the standard deviation and perhaps the mean of the required gain distribution.

Some MBBs have a very large proportion of government controlled loans, especially in connection to loans issued by social housing associations. The implication on this was seen December 16, 1991, when the Ministry of Housing announced the simultaneous prepayment of all loans issued in 12% bond series. In the present model this kind of behaviour could be modelled as a group of borrowers with a very small σ, but the mean of the distribution is unknown a priori. Other examples exist in which a single borrower controls a large proportion of an outstanding loan volume and these cases should be singled out if possible and handled with care.

To summarize we have set up a model in which the rate of prepayments depends on the observed gain from prepayment through a continuous distribution function. As opposed to the American option model this results in smooth behaviour of prices and the model could easily cope with the uncertainty and perhaps even irrationality, which characterizes actual prepayment behaviour.

6.3 Price-behaviour

The pricing model for the MBB is complicated and we are not aware of any closed form solutions, which would allow a formal mathematical analysis. This section therefore contains a sensitivity analysis of the model to assess the impact of different parameters on the price of MBBs. Most of the interactions are perhaps quite obvious from a theoretical viewpoint, but they are stated here for reference. Our basic scenario and the choice of bonds is similar to the current market situation and numerical magnitudes should therefore be of interest. The prepayment model resembles the paper by Jakobsen(1992), but that paper used a binomial process for the yield to maturity. The present papers use of an arbitrage-free term structure model makes it possible to study how the shape of the initial term-structure affects prices.

To visualize the wealth of interactions we have choosen three 20 year annuity bonds with quarterly payments and coupon rates of 9, 10 and 12%. These values represent some of the largest MBBs in the Danish market. To study the impact of the cash-loan system a 20 year 12% annuity bond with a cash-loan rate of 14% is included[59].

The basic term structure scenario assumes a flat yield curve at 10% annual compounded, future short term interest rate volatility of 10% and 8 steps per year. Borrowers are assumed to have a tax-rate of 50%, a 3 months term of notice and a cost rate of 1%. Their prepayment behaviour is described by a constant mean required after-tax gain of 12% with a standard deviation of 3%. Starting values for the four bonds are shown in table 6.3. The 9% mortgages have a negative gain from prepayment and prepayments at the initial yield of 10% only occurs in the two 12% bonds. With reference to option valuation terminology we shall refer to the 12% bond as being in-prepayment, the 10% and the 12/14% is at-prepayment and the 9% is out-of-prepayment.

Table 6.3: *Initial values for sensitivity analysis.*

Mortgage holder:	9%	10%	12%	12/14%
After-tax value	97.16	101.80	111.52	107.88
Prepayment cost	100.90	101.02	101.27	101.27
Prepayment gain (%)	-3.85	0.76	9.20	6.13
Prepayment rate (%)	0.00	0.00	17.52	2.53
Bond investor:	**9%**	**10%**	**12%**	**12/14%**
Yield of non-call	10.00	10.00	10.00	10.00
Price of non-call	95.55	102.48	116.90	116.90
Prepayment value	99.84	100.09	100.57	100.57
Price of MBB	94.27	99.00	102.23	105.29

59 The cash-loan system refers to bonds issued before 1986 cf. section 5.3. A typical cash-loan would have semiannual payments, 5 months term of notice and a maturity less than 20 years but to compare the results we have assumed the same basic parameters as the three bond-loans.

Figure 6.3 gives the prices as a function of the market rate of interest. At a yield of 10% the two 12% bonds prices are at their maximum. The 12% bond-loan will not reach much above 102 while the 12/14% cash-loan goes 3 price points higher because the tax-advantages of the cash-loan system reduces the borrowers gain from prepayment. At higher interest rates the two 12% bonds converge. Should any 12% bonds still be on the market, when interest rates go below 8%, their prepayment rates would be close to 100% ,and their prices lower than prices of 9 and 10% bonds.

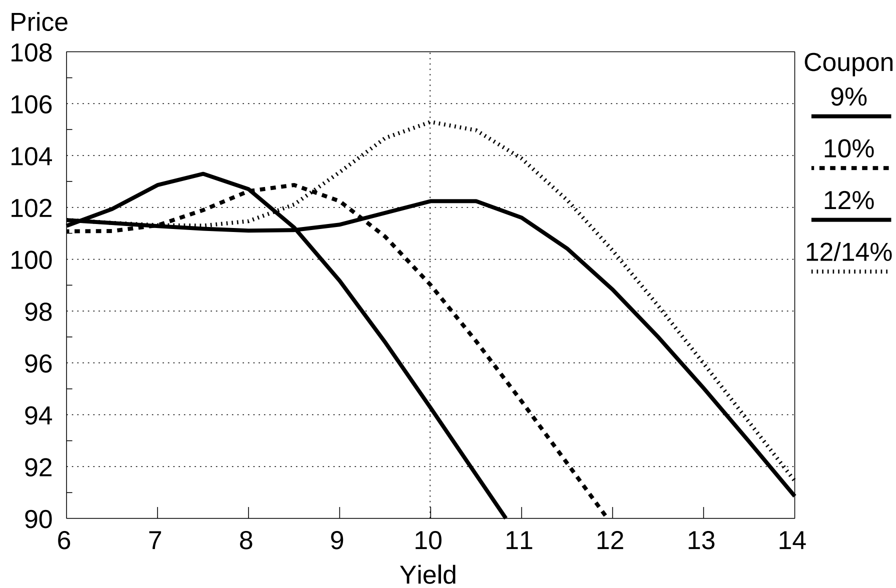

Figure 6.3: Prices of four 20 year MBBs as a function of the noncallable market rate of interest.

The following sensitivity plots show the prices of the MBBs as functions of different parameters. For most plots the price of the similar non-callable bond given in table 6.3 stays constant. If the prepayment option gets far out of the money, the MBB-price reaches this upper limit. The corresponding prepayment value will likewise act as a lower boundary for very high rates of prepayment.

The assumption on borrowers behaviour is given by the mean and standard deviation of required gain. Generally a higher mean required gain lowers prepayment rates and the MBB-price increases. An increase in average gain has most effect on the 12% bonds, due to high value of their prepayment option. Bonds, in which the prepayment option is out-of-the money, like the 9 and 10% bonds, are only affected to a lesser degree. At average gains of 17% the 9 and 10% bonds trade close to their non-call values.

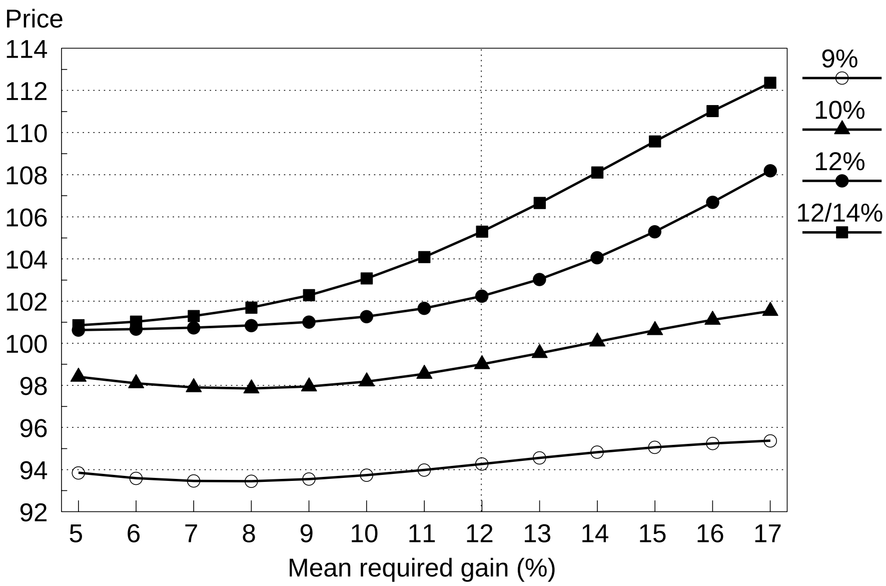

Figure 6.4: Prices of four MBBs for different values of mean required gain.

For very low values of average gain, there is a positive prepayment rate, even for the low coupon bonds. For 9 and 10% bonds prices are below par and the prospect of prepayment now or in the immidiate future will increase rather than decrease values. This price behaviour is perhaps a weakness of the model. In practice no mortgage owner would prepay at par, when he could purchase bonds below par and cancel the loan through the delivery option as discussed in section 5.4[60]. For the 12% bonds, prepayment at negative gains is a definite posibility, especially for the liquidity reasons discussed in the previous section.

The standard deviation of required gain measures the degree of heterogeneity among otherwise similar borrowers. Reasons for individual borrowers to differ on required gain was given in the previous section. As heterogeneity increases more borrowers will prepay at higher interest rates, and this will ceteris paribus decrease the value of the MBB as shown in figure 6.5. The change in σ has most effect on the 12/14% cash-loan, which was close to prepayment. The 12% bond-loan with high prepayment rates is less influenced, as are the 9% and 10% bonds, which are rather far from prepayment.

60 The prepayment option could be incorporated in the model along the lines of section 5.4, but given reasonable values of the prepayment function, nothing would be gained by this complication.

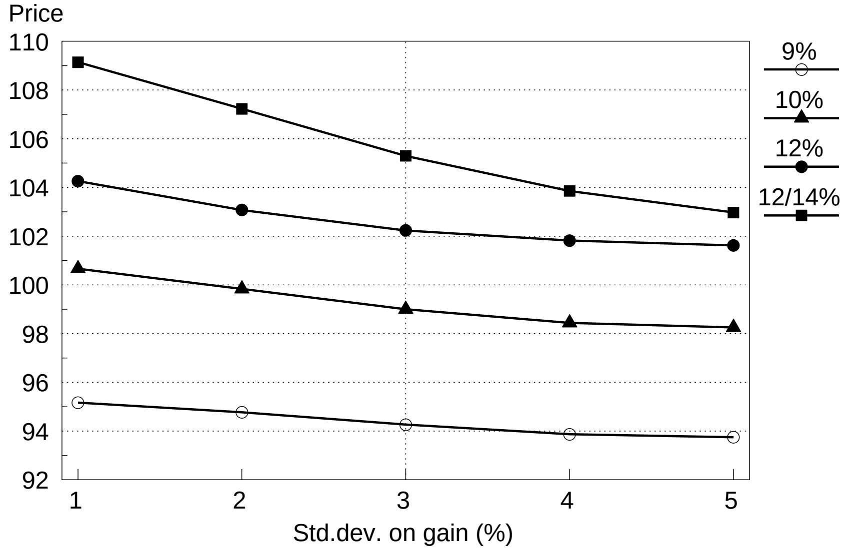

Figure 6.5: Prices of four different MBBs as a function of the standard deviation for required gain.

The sensitivity to σ was computed with the two 12% bonds at their maximum values. As interest rates falls below 10%, 12% MBB-prices falls and one could wonder, if the impact of σ reverses, because a smaller σ leads to a steeper descent of the MBB-price.

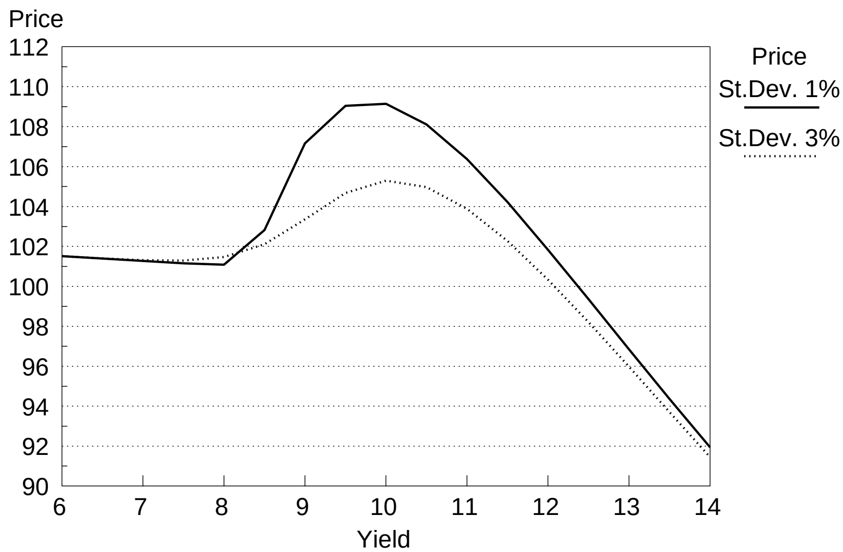

Figure 6.6: Prices as a function of market yield for different values of σ.

To study the sign of $\partial V/\partial\sigma$ for all levels of interest rates, we have calculated prices of the 12/14% for σ equal to 1% and 3% respectively. Barring a minor range around 8% yield it is seen from figure 6.6 that an increase in σ uniformly leads to a fall in the MBB-price. The steeper descent of the 1%-price is matched by a higher value for its maximum. Prices thus increase ceteris paribus with a decrease in heterogeneity.

Some reservations should be mentioned. If the mean required gain g^* is unknown a priori, as in the above-mentioned case of the Ministry of Housings collective prepayment of all 12% loans, then an alternative model - perhaps like the American option model - would be more appropriate. If secondly borrowers are divided into separate identifiable groups, then the assumption of a single normal distribution should be replaced with the grouping approach of section 6.5.

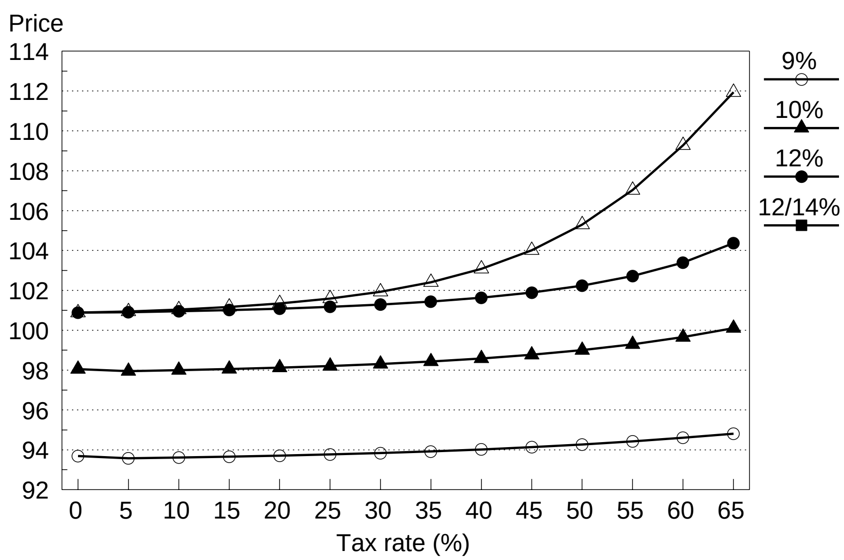

Figure 6.7: The impact of the tax-rate on MBB-prices.

In section 5.3 it was argued that an increase in the marginal tax-rate of borrowers should lead to a fall in the critical yield, due to the tax-advantages lost by prepaying. In the current model taxes affect prepayment rates through a reduced gain from prepaying, and figure 6.7 shows the resulting impact on prices for tax-rates in the range from zero to 65%[61]. Most borrowers are either corporations with tax-rates around 35% or private households with tax-rates between 50 and 60%[62]. Below 55% the exact tax-status of the borrowers seems to matter rather little at least for the three bond-loans. The 12/14% cash-loan enjoys a special tax-advantage and a knowledge of the exact tax-status of borrowers will be important.

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61 Foreign readers should note that we are not trying to prove a limiting result. A small group of persons with large personal wealth face marginal tax rates above 68% on interest payments.

62 The low precision on these figures reflects that tax-rates and tax-systems for corporations have been changed several times within the last few years. Marginal tax rates for private households varies cross country and it differs according to the sign of personal net wealth.

Figure 6.8 displays the MBB-value as a function of time to maturity. Prices of corresponding non-callable bonds are included for reference. With a flat yield curve of 10% the prices of non-callable bonds are dragged towards 100 as time to maturity decreases. The MBB-prices behave rather differently. If borrowers demand a 12% average gain irrespective of the time to maturity, then the prices of 30 year bonds will be highly affected by prepayments, while 10 year bonds are close to the corresponding prices of the non-callable.

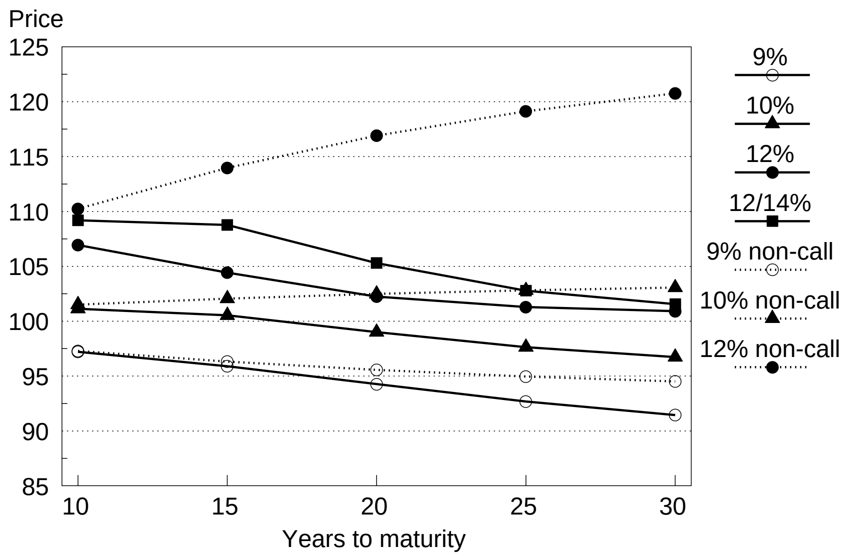

Figure 6.8: Prices as a function of time to maturity assuming a maturity independent required gain of 12%.

If borrowers receive proper advise, we would expect a lower mean required gain for the shorter maturities. This behaviour is implicitly assumed in the American option model, where the borrower make a rational decision comparing current gain from prepayment with possible future gain. For comparison we have calculated the same set of values for the American option model as shown in figure 6.9 below.

Assuming rational behaviour is seen to have a dramatic effect. In the AO-model MBB-prices are almost constant accros maturities and the prepayment option is valuable even for short maturities. To see, if pricing results of the American option model could be replicated by the RG-model, the model is changed to allow for a maturity dependent mean required gain (MDRG). The MDRG-model assumes that the mean and standard deviation of the required gain distribution decreases linearly with remaining time to maturity ending at zero at maturity. The results of the MDRG-specification is compared to the AO and the RG prices in table 6.4. In the MDRG-calculation a 12% gain and a 3% standard deviation with 20 years remaining time to maturity is chosen to compare easily with the fixed distribution of the

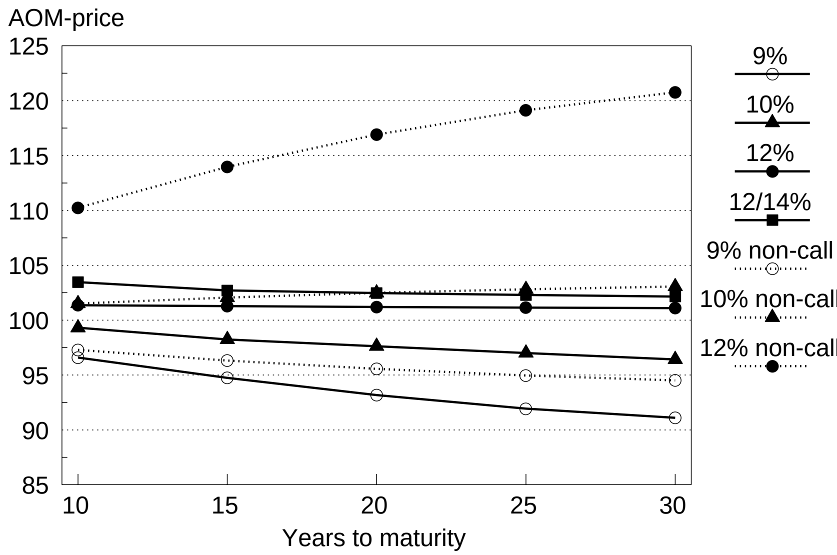

Figure 6.9: Prices implied by the American option model as a function of time to maturity.

RG-model. In the MDRG-model a 30 year bond will thus start at 18% gain with a 4.5% standard deviation. With 10 years remaining the figures will be 6% and 1.5% respectively.

For the 20 year bonds the MDRG-model has the same initial gain distribution as the RG-model, but the assumed fall in mean required gain increases future prepayment rates and the MDRG-prices are therefore lower. The difference is rather small, however, and typically below one price point. The main difference between the MDRG- and the RG-model occurs for bonds of longer or shorter maturities. Here the MDRG-model leads to a price behaviour which closely resembles the AO-model[63]. We therefore conclude that a very simple change in the specification of the required gain model could cope with one of the most important lessons learned from the study of rational borrowers behaviour. In this respect the required gain model allows for "rational" as well as irrational behaviour. What borrowers *actually* do is an empirical question cf. the discussion in chapter 7.

The remainder of this chapter sticks to the RG-model with a fixed distribution of mean required gain, but the maturity dependent version will be used in the next chapter.

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63 A closer match to AO-prices could be obtained by the use of a lower required gain at 20 years. But the relevant prices to match will of course be an empirical question. The strength of the MDRG-model is that the parameters can be chosen to match observed prices and prepayment rates.

Table 6.4: *Comparison of pricing results as a function of time to maturity for different models.*

	Years to maturity	Fixed mean required gain	American option model	Maturity dependent mean required gain	Price of non-callable bond
9% annuity	10	97.21	96.59	96.87	97.28
	15	95.89	94.73	95.13	96.32
	20	94.27	93.17	93.68	95.55
	25	92.68	91.93	92.51	94.96
	30	91.44	91.10	91.60	94.50
10% annuity	10	101.13	99.31	99.63	101.51
	15	100.53	98.23	98.97	102.05
	20	99.00	97.62	98.24	102.48
	25	97.63	97.01	97.62	102.81
	30	96.73	96.42	97.16	103.06
12% annuity	10	106.94	101.36	101.52	110.24
	15	104.42	101.27	101.79	113.97
	20	102.23	101.20	101.93	116.90
	25	101.28	101.14	102.10	119.13
	30	100.91	101.10	102.34	120.76
12/14% annuity	10	109.19	103.46	104.12	110.24
	15	108.77	102.70	104.60	113.97
	20	105.29	102.46	104.19	116.90
	25	102.78	102.28	103.81	119.13
	30	101.55	102.15	103.65	120.76

An increase in the future volatility of short term interest rates increases the future range of pre- and after-tax prices of the non-callable bond. The likelihood of prepayment as well as the loss to investors in case of prepayment is therefore increased. This will lead to a fall in MBB-prices as shown in figure 6.10. The effect is largest for at-prepayment bonds like the 10% and 12/14%. For the in-prepayment 12%

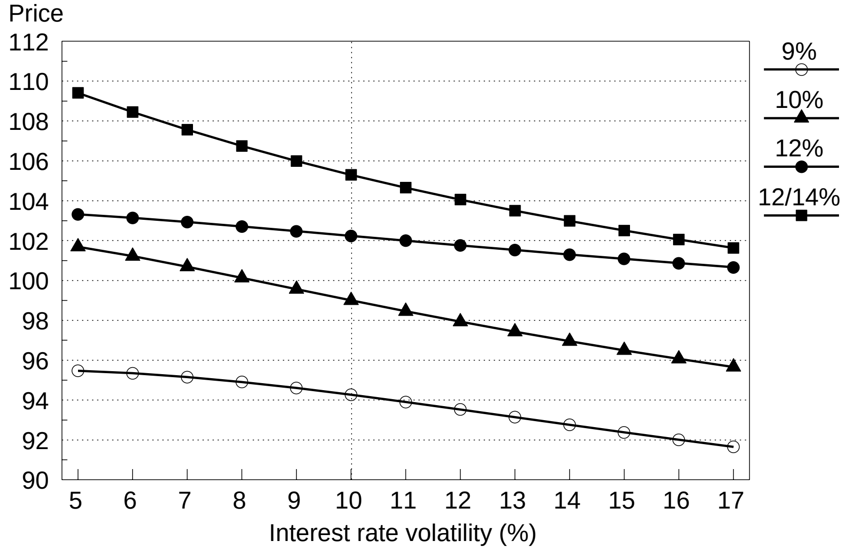

Figure 6.10: Prices of mortgage backed bonds as a function of interest rate volatility.

bond most borrowers will prepay in the near future irrespective of future volatility, and the effect is thus smaller. The same goes for the out-of-prepayment 9% bond, for which prepayment is still a distant possibility.

The volatility results from the RG-model are quite similar to the results expected from an American option model. Note however that mean required gain is unaffected by the change in volatility. In the AO-model an increased volatility leads to a higher future value of the prepayment option, which implies that borrowers would prepay at a lower critical yield. To incorporate this kind of rationality in the RG-model, higher values of mean required gain should be used for higher volatilities. Rational adjustment would thus once again dampen the price effect of volatility changes.

An interesting feature unique to the arbitrage-free term structure models is the direct way, in which the initial term-structure determines the stochastic process of future interest rates. This allows us to study how MBB-prices are affected by the shape of the initial term-structure.

Consider a range of linear initial yield curves with slopes between -0.25% and 0.25% corresponding to a maximum spread between the 0 and 20 year yield of ±5 percentage points. For each choice of slope an intercept value is computed, which ensures a 10% yield to maturity of a 20 year annuity bond. Corresponding slope and intercept values are given in table 6.5 below. By construction a change in slope has no effect on non-callable bond prices, but there may be an effect on MBB-prices as illustrated in figure 6.11. Besides the four bonds used above we have calculated results for a 10% bond with term of notice and tax-rate equal to zero.

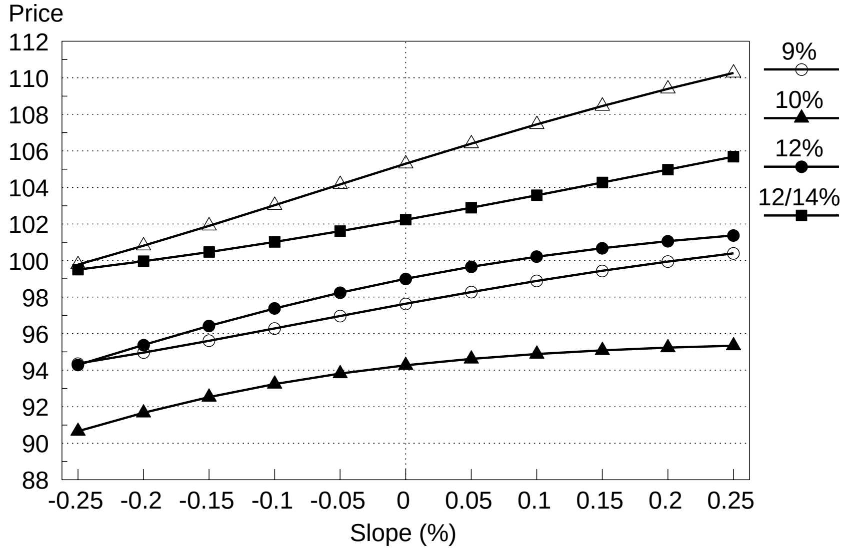

Figure 6.11: The MBB-price effect of changes in term structure slope for a constant non-callable yield to maturity

Despite the constant yield to maturity of non-callable bonds there is a clear positive dependency between MBB-prices and the slope of the initial term structure. An increase in spread between long and short term yields decreases the value of the prepayment option and vice versa.

Table 6.5: *The dependency of term structure shape on MBB prices.*

Term structure		Tax 0%, coupon	20 year annuity bonds, tax-rate 50% coupon rate:			
Slope	Inter-cept	10%	9%	10%	12%	12/14%
-0.25	12.85	94.36	90.65	94.29	99.50	99.78
-0.20	12.27	94.96	91.66	95.38	99.96	100.81
-0.15	11.69	95.61	92.52	96.42	100.47	101.90
-0.10	11.12	96.28	93.24	97.38	101.02	103.03
-0.05	10.56	96.96	93.81	98.24	101.61	104.17
0.00	10.00	97.63	94.27	99.00	102.23	105.29
0.05	9.45	98.28	94.61	99.65	102.89	106.39
0.10	8.91	98.88	94.88	100.21	103.57	107.45
0.15	8.37	99.44	95.08	100.67	104.27	108.45
0.20	7.84	99.94	95.23	101.06	104.97	109.39
0.25	7.32	100.39	95.34	101.37	105.68	110.26

In section 4.3 we showed that the initial after-tax value of a non-callable annuity bond was almost unaffected by a change in slope, provided yield to maturity stayed constant accross term structures. The prepayment costs are not constant however, and a simple explanation of the slope dependency could therefore be that the term of notice causes prepayment costs to rise with a fall in short term interest rates. This reduces borrowers gain from prepayment and increases the value of the MBB in case of prepayment. Both effects should tend to increase MBB-price with increasing slope. But as seen from the table the slope effect is present even for the bond with a zero term of notice. For this bond initial prepayment rates are entirely unaffected by the change in slope.

A heuristic explanation could be that callable bonds are ’shorter’ than non-callable bonds. An increase in slope increases prices of short term bonds relative to long term bonds which explains the slope dependency. The heuristic argument implicitly assumes that average maturity of the MBB stays constant across term structures.

For a full explanation one must consider the dynamics of the binomial term structure process. Some evidence is given in figure 6.12 for initial slopes of -0.25, 0 and 0.25%. Figure 6.12-a shows the initial term structures, and as argued above the initial prepayment rates are almost equal for all three cases. But future prepayment rates are highly dependent on the initial slope. An upward sloping initial term structure implies that future yields should on average be above current yields. This leads to lower future prepayment rates. Investors include these expectations in their valuation of the MBB, which explains why out-of-prepayment MBBs converge toward their non-callable values as the slope increases. In-prepayment bonds like the 12% will still be sold at a discount due to high initial prepayment rates.

A downward sloping initial yield curve will likewise imply average future yields below current yields, and therefore the investors expect higher prepayment rates relative to our basic scenario with a flat yield curve. Figure 6.12 (b)-(d) shows the 5 year forward term structures at the lower quartile, median and upper quartile of the short term interest rate distribution. In all three cases forward term structures for the upward sloping yield curve lie above the ones for the downward sloping yield curve.

A constant mean required gain was used throughout the example. In the AO-model an increase in slope reduces the future value of the prepayment option and borrowers react by prepaying earlier. If we want this level of rationality in the RG-model, a

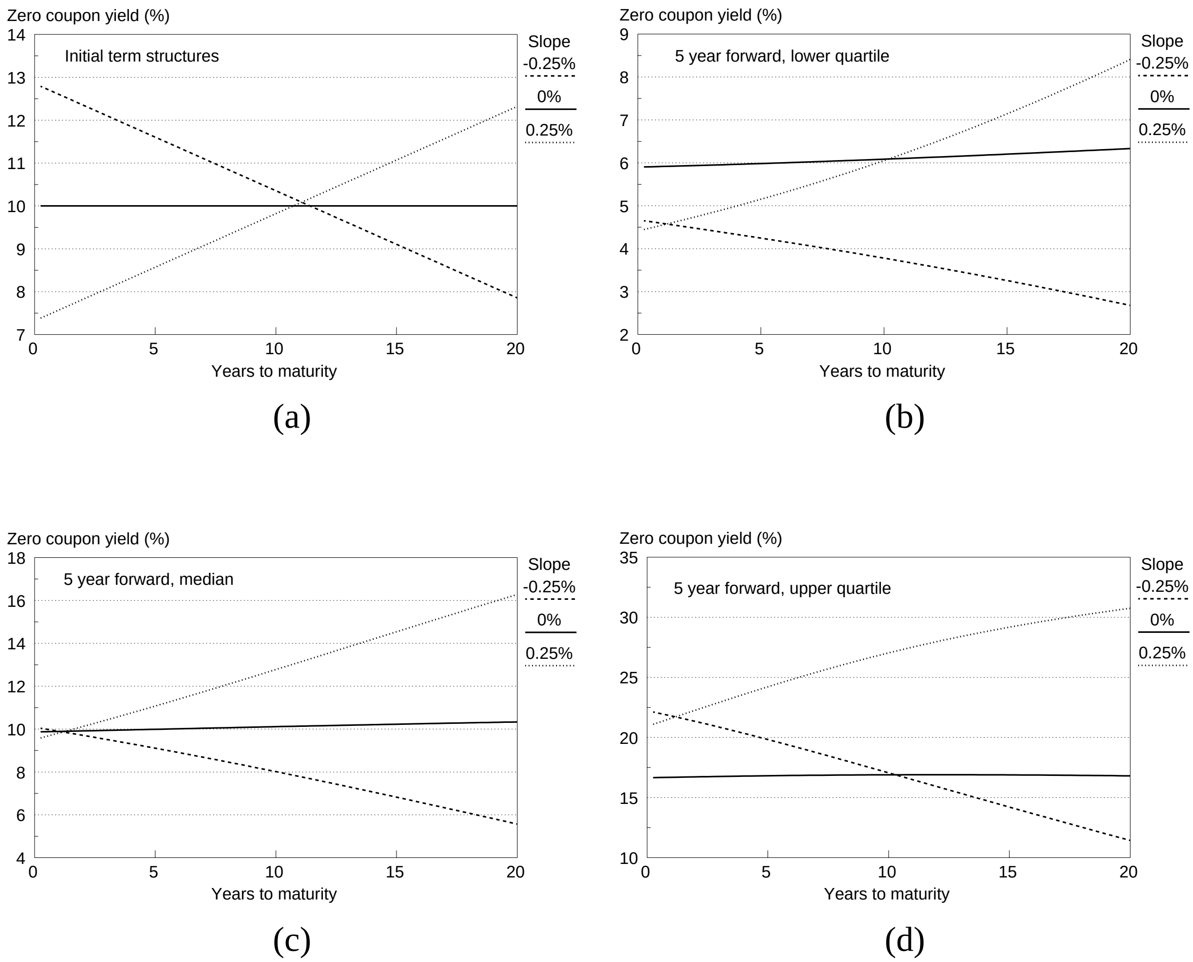

Figure 6.12: (a) Initial yield curves of different slope. (b-d) Forward yield curves at lower quartile, median and upper quartile of short term interest rate distribution after 5 years.

higher mean required gain should be used for an upward sloping yield curve and vice versa. Introducing rational behaviour would thus make MBB prices less dependent on initial slope.

To summarize we have presented a sensitivity analysis of the mean required gain model. It was shown how MBB-prices depended on the yield level, the required gain distribution, borrowers tax rate, remaining time to maturity, and the volatility and slope of the initial yield curve. The results were easily explainable, and hopefully they provided a deeper insight into the pricing model. References to the American option model was given throughout, and in general it was argued that the MBB-prices of the

AO-model was less sensitive to parameter changes, because the change in rational borrowers prepayment behaviour would typically counteract the change in the investors valuation of future cash flows.

The choice between the RG-model and the AO-model is not an obvious one. From a computational point of view the two models are equivalent, but the AO-model suffers from discontinuity problems as shown in section 5.5. On the other hand it provides a full endogenuous determination of prepayment behaviour consistent with rational behaviour. The RG-model allows the description of rational as well as irrational behaviour, and one can easily analyse, how observed prepayment behaviour affects prices. On the negative side the prepayment distribution must be user-supplied, and to incorporate some level of rationality, the distribution should at least be made dependent on time to maturity.

6.4 Duration measures

The fluctuation of interest rates is the primary source of risk for fixed payment bonds. Although MBB-pricing must cope with a vast amount of prepayment uncertainty as well, interest rate risk measures like duration and convexity will still be valuable for the investor. This section starts with a short review of general duration measures for non-callable fixed-payment bonds. In the context of an arbitrage-free term structure model, these measures are easily extended to bonds with embedded options, like the MBBs. In-prepayment MBBs are often viewed as a direct substitute for short term non-callable bonds, and we compare the return behaviour of these bonds to illustrate possible pitfalls.

Let P denote price and y the continuously compounded[64] yield to maturity of a non-callable bond with payments b_t at time $t = 1, \ldots, N$. The Macaulay/Redington (MR) duration measure is defined as

(6.2)
$$D_{MR} = \left[\sum_{t=1}^{N} t \cdot b_t e^{-y \cdot t} \right] / P$$

The MR-duration measure is often viewed as a weighted average maturity of the bond, but its practical value comes from the relation

64 Duration and convexity could be defined using discrete compounding as well but continuously compounding simplifies most formulas.

(6.3) $$\frac{\partial P/\partial y}{P} = -D_{MR}$$

which states that duration measures the security's relative price sensitivity to yield to maturity. As discussed in e.g. Brennan and Schwartz (1982) the widespread use of MR-duration as a risk measure implicitly assumes a single factor model with parallel shifts in a flat yield curve as the single source of interest rate risk[65].

Following the paper by Fisher and Weil (1971), Bierwag and Kaufman(1977), Khang(1979) and others have proposed more general duration measures, involving additive and multiplicative shifts in non-flat yield curves. In the present analysis we shall confine ourselves to a subset of these duration measures.

We assume that changes in t-period zero coupon yield, R_t, can be described by a function of time to maturity $H(t)$ multiplied by a scalar h. Each value of h implies a new yield curve $R_t(h) = R_t + hH(t)$. The scalar h can be viewed as the degree of change in the direction given by $H(t)$. The bondprice P will change to $P^H(h) = \sum b_t \exp(-t \cdot (R_t + h \cdot H(t))$. For each choice of shift-function, $H(t)$, the corresponding $H(t)$-duration measure is defined by

(6.4) $$D_H = \frac{P_H(0)}{P} = \sum_{t=1}^{N} t \cdot H(t) \cdot b_t \cdot e^{-R_t t}/P$$

where $P_H(h) \equiv \partial P^H(h)/\partial h$ denotes the partial derivative of $P^H(h)$ with respect to h[66]. D_H is thus the relative sensitivity of price to an infinitisimal yield curve shift of type H(t). We shall refer to these general duration measures as generalized Fisher-Weil durations to emphasise their dependency on the initial zero coupon yield curve.

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65 Ingersoll, Skelton and Weil(1978) and Cox, Ingersoll and Ross(1979) have shown that the different duration measures implies different restrictions on the possible stochastic behaviour of interest rates.

66 The cumbersome use of the scalar h is needed to allow a formalization of a partial derivative with respect to the shift-function H(t). A similar approach can be found in Nielsen(1987).

Setting $H(t) \equiv 1$ corresponds to the parallel or level shift in the initial yield curve introduced by Fisher and Weil(1971) . From (6.4) we get the level-duration measure D_L as[67]

(6.5) $$D_L = \sum_{t=1}^{N} t \cdot b_t \cdot e^{-R_t \cdot t} / P$$

Likewise a change in slope corresponds to $H(t) \equiv t$ which leads to slope-duration measure, D_S, given by

(6.6) $$D_S = \sum_{t=1}^{N} t^2 \cdot b_t \cdot e^{-R_t \cdot t} / P$$

Duration measures for non-linear shift can be found by a simple redefinition of $H(t)$. Setting $H(t) \equiv \alpha + \beta t$ the analogous duration measure can be found by

(6.7) $$D_{\alpha,\beta} = \alpha D_L + \beta D_S$$

Knowledge of D_L and D_S can thus be used to compute the sensitivity of bond value to any linear shift in the initial yield curve. For discrete linear shifts one can use the approximating formula

(6.8) $$\Delta P / P \quad \approx \quad D_L \Delta L + D_S \Delta S$$

where ΔL and ΔS is the assumed change in level and slope respectively.

To extend the Macaulay/Redington duration measure to a portfolio of bonds one must in principle recompute the duration for the full aggregate cash flow of the portfolio due to the well known non-additiveness of yield-to-maturity. The generalized FW-duration measures are far simpler and the duration, D_H^p, of a portfolio can be found by a simple weighted average of individual duration measures D_H^i

(6.9) $$D_H^p = \sum x_i \cdot D_H^i$$

[67] In the following we use subscript L (level) and S (slope) to indicate $H(t) \equiv 1$ and $H(t) \equiv t$ respectively.

where x_i denotes the value-share of bond i in total portfolio value.

Duration measures can be seen as a first order approximation to the underlying price/yield relationship. As such it applies only to small shifts in the yield curve. To analyse somewhat larger shifts one could use a second-order approximation. Assuming $H(t) \equiv 1$, corresponding to a parallel shift in the initial yield curve, we get the following second order Taylor approximation to the relative change in price from a level change of ΔL:

(6.10) $$\frac{\Delta P}{P} \approx -D_L \cdot \Delta L + \frac{P_{LL}}{2 \cdot P} \cdot \Delta L^2 \equiv -D_L \cdot \Delta L + \frac{C_L}{2} \cdot \Delta L^2$$

P_{LL} denotes the second partial derivative of price with respect to a level shift and $C_L \equiv P_{LL}/P$ is called the *convexity* of the bond. Equation (6.10) shows that if we compare two bonds with equal level-duration, then the bond with the highest convexity will gain most from a fall in the level of interest and it will loose less from an increase in the level of interest. This apparently attractive property has lead many bond investors to prefer high convexity bonds.

By direct calculation of the second partial derivative the convexity measure can be expressed by the well known formula

(6.11) $$C_L = \left(\sum_{t=1}^{N} t^2 \cdot b_t \cdot e^{-R_t t} \right) / P$$

Comparing to equation (6.6) we get the interesting result that slope-duration equals convexity for fixed cash-flow bonds. Investors pursuing gains from convexity will therefore maximize their exposure to losses from an upward shift in yield curve slope[68].

68 The exact equivalence is due to the use of continuously compounding. For discrete compounding small deviations occur. To give a heuristic explanation duration decreases with a rise in yield level because the value of long term payments decrease relative to short term payments. Convexity measures this rate of change in duration through the relationship $D_{LL}/P = D_L^2 - C_L$. Bonds with high convexity have a large dispersion of payments which results in a large change in duration. But these bonds will also be most affected by a change in slope.

For fixed cash-flow bonds the generalized duration measures can be found from the above-mentioned formulas. For MBBs and other securities with interest dependent cash-flows duration measures must be calculated by numerical methods. We exploit the fact that arbitrage-free term structure models price bonds and related instruments relative to the initial term structure. To obtain a generalized duration measure the model is recalibrated for a shifted yield curve $R_t + h \cdot H(t)$ and a new valuation is done resulting in the 'shifted' price $P^H(h)$[69]. The approximated duration measure is given by

(6.12) $$D_H = \frac{(P - P^H(h))}{h \cdot P}$$

Convexity measures requires the numerical calculation of a second order partial derivative. The initial yield curve is shifted downwards by -*h* and the approximated convexity measure is given by

(6.13) $$C_H = \frac{P^H(h) + P^H(-h) - 2 \cdot P}{h^2 \cdot P}$$

The following figures show how level-duration, slope-duration and level-convexity depends on the level of an initial flat yield curve. The values are computed for the four MBBs analysed in the previous section as well as a 20 year non-callable annuity bond. As expected the non-callable bonds show positive convexity throughout and level- as well as slope-duration increases with a falling level of interest.

The MBB duration measures display a much more complex pattern. At high rates of interest the MBBs behave like their underlying non-callable bond. But as interest rates fall the MBB-duration measures fall below durations for the non-callable bond and very soon we get a decrease in price sensitivity as yields decrease. Interestingly this happens long before prepayment becomes imminent. The 12% and the 12/14% bond with the highest prepayment risk will of course be the first to diverge from the non-callable bond.

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[69] For each duration measure we need a full calibration of the shifted lattice as discussed in chapter 3 plus a 'shifted' valuation for each security. The calibration procedure is however common to all securities and the resulting parameters for the shifted short rate process can be stored and reused. When calculations are done for several securities only the valuation step counts, but still computation times triples if two duration measures is needed on top of the bond valuation itself. After som experimentation we have set $h = 0.001$ which corresponds to a 10 basis point level-shift (0.1 percentage point). Numerical instability of convexity measures occurs when $h = 0.0001$.

Figure 6.13:

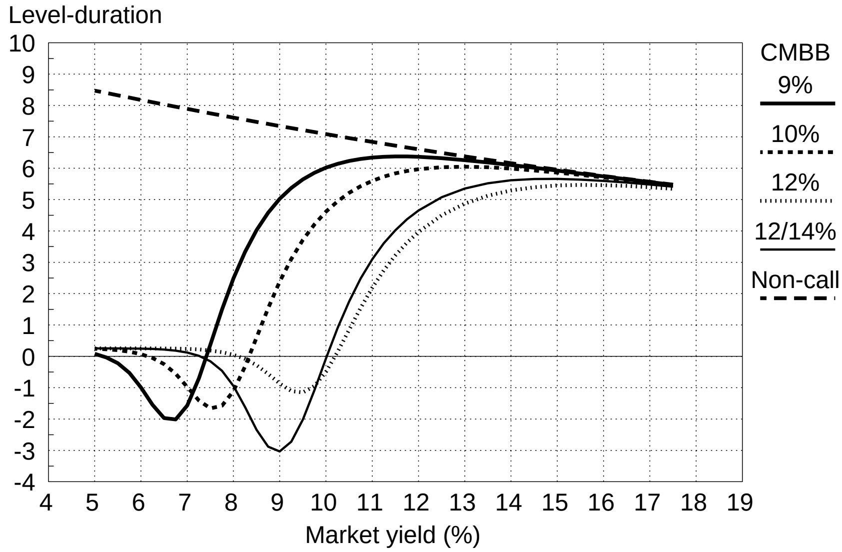
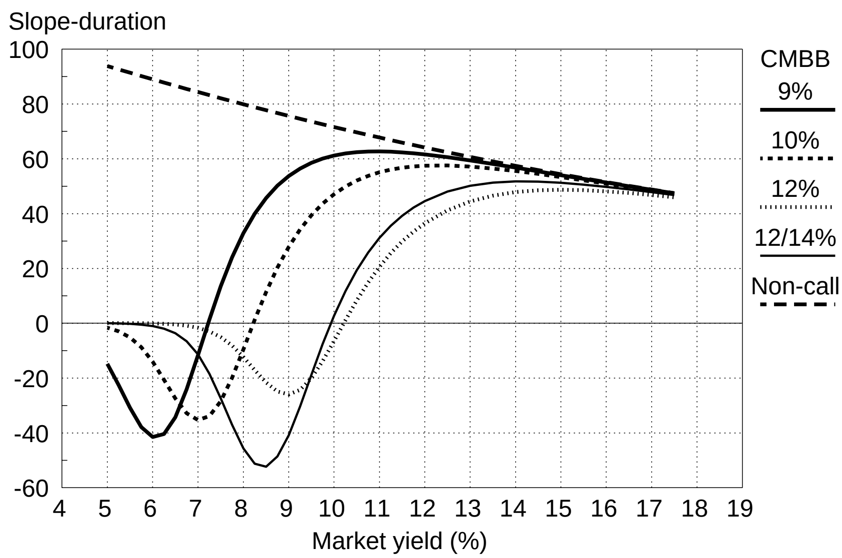


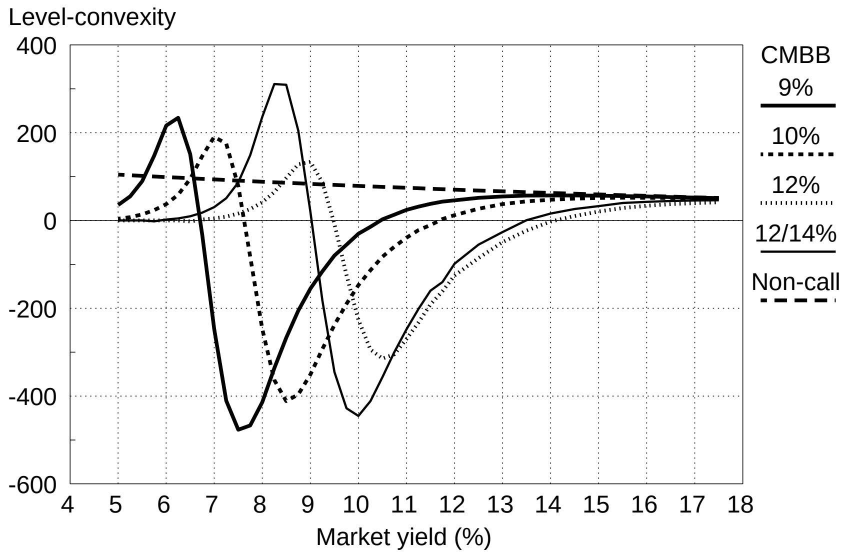
Figure 6.14:



As seen in the previous section MBB-prices reach a maximum which corresponds to a level-duration of zero. Below this point there is a region of negative duration with MBB-prices moving against non-callable bond prices. Finally, at very low levels of interest rates the MBBs reach full prepayment and level-duration equals the time to the first settlement date.

Duration measures become very volatile around the maximum MBB-price and given the uncertainty on prepayment behaviour they should be used for pedagogical purposes mostly[70]. The duration measures for at-prepayment or out-of-prepayment MBBs are probably much more useful especially to owners of large portfolios who need a measure of aggregate yield sensitivity.

Figure 6.15:



The plot of level-convexity reflects the high volatility of level-duration to an even higher degree. The MBBs generally have a lower convexity than their non-callable counterpart. For large regions of yield levels the MBBs have the undesirable property of negative convexity, which imply that duration decreases when interest rates falls and duration increases with a rise in interest rates. Capital gain will thus be smaller and capital losses larger for a MBB relative to a non-callable bond of similar duration.

As a further implication of the interest rate dependency of MBB cash flows it is seen that no equality between slope-duration and level-convexity exists for MBBs except at very high levels of yield[71].

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70 The values were calculated with prepayment rates described by a normal distribution function. A right-skewed distribution would result in a slower increase toward full prepayment and perhaps a less volatile behaviour of in-prepayment duration and convexity measures.

71 The generalized Fisher-Weil durations applies an exogenously specified shift to the current yield curve. These shifts may be different from the arbitrage-free shifts allowed by the assumed stochastic process for the short term interest rate. Instead one could calculate the sensitivity of the individual securities to changes in the short term rate of interest. Let $V(n,s)$ denote the price of a security at date-event (n,s) and let $r(n,s)$ be the corresponding annualized short term interest rate. The arbitrage-free (AF) duration measure is defined as:

$$D_A(n,s) = \left[\frac{V(n+1,s) - V(n+1,s+1)}{r(n+1,s+1) - r(n+1,s)} \right] / V(n,s)$$

AF-duration can be viewed as a special kind of generalized Fisher-Weil duration with an arbitrage-

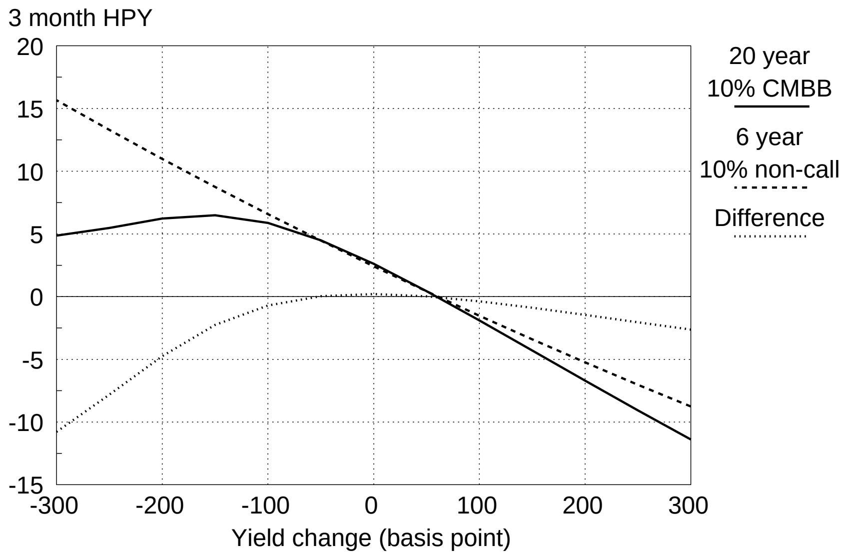

Figure 6.16:

The analysis of MBB duration measures can be summarized by a discussion of holding period return (HPR). Holding period return is defined as the return of a bond investment over a given period of say 3 months. At the beginning of the period the bond is bought at its current market price plus accrued interest. After 3 months the bond is sold and the total end-of-period value equals the sum of market value and accrued interest as well as the accumulated value of any coupon payments and repayments due in the period. For in-prepayment MBBs the repayment portion could be rather large due to high prepayment rates. End-of-period market value will of course depend on end-of-period yield curves.

Figure 6.16 shows the 3 month HPR for the 20 year 10% annuity bond. We assume a 10% flat initial yield curve and the HPR is calculated for different levels of end-of-period yields. The HPR for the MBB is compared to the HPR for a 6 year 10% non-callable ordinary bond with quarterly payments. Initially the two bonds have almost identical level-duration with D_L equal to 4.62 and 4.58 respectively. Level-convexity differ sharply with C_L equal to -147.33 for the MBB and 24.90 for the non-callable bond. We assume both bonds to be bought and sold at their theoretical values[72].

It is seen that for small changes in interest rates the two bonds have almost similar performance. But for changes above ±50 basis points the MBB underperforms the non-callable bond due to its undesirable change in duration. When rates increases the

free shift endogenously determined as the spread between up- and down-state yield curves.

[72] A similar analysis for callable corporate bonds can be found in Latainer and Jacob (1985).

MBB gets longer and vice versa. For long-term investments most investors would probably prefer the non-callable bond, but short-term investors would be indifferent, provided that their portfolios are frequently rematched.

Figure 6.17:

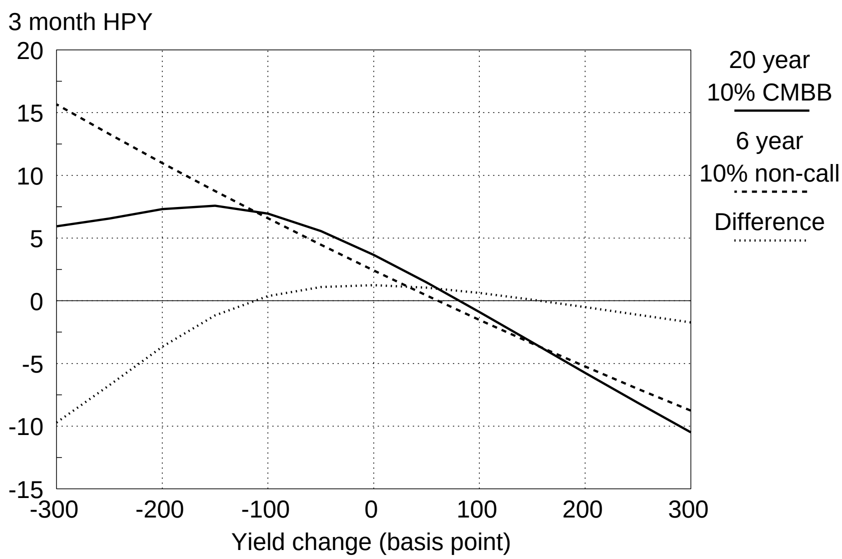

Figure 6.17 shows a similar plot, but now the MBB is assumed to be priced 100 basis points below its theoretical value. The bond returns to its theoretical value during the period. The MBB now outperforms the non-callable bond for a wider range of yield changes and it will be very attractive to an investor who expects a stable interest rate regime.

6.5 Different groups of borrowers

Up til now we have assumed a single group of borrowers with a normal distributed required gain. In practice borrowers may be divided into subgroups, each characterized by a separate distribution. In this section we will discuss the implications of grouping. Data on an 11% annuity bond maturing in 2010 will be used to illustrate the analysis, but no attempt has been made to fit the model in any way.

As seen from the sensitivity plots of section 6.3 several parameters of the model may be used to stratify borrowers into different subgroups. The tax-rate is an important determinant especially for cash-loans, cf. figure 6.20[73]. Differences in mean required gain and prepayment costs could be used as grouping parameters as well. Parameters like coupon rate, terms of notice, years-to-maturity[74] etc. is common to all borrowers in a single MBB and no subgrouping is necessary. Other important price-determinants like initial term structure and future short term volatility is common to all MBBs.

In the following example the individual mortgage owners are divided into firms and households. Firms have a marginal tax rate of 38%[75] and a 2% cost-rate. The mean required gain for firms is assumed to be 8% with a standard deviation of 2%. The marginal tax-rate for households is 56%. Households face costs at 3% and we assume a mean required gain of 10% with a standard deviation of 3%. The choice of parameters reflects that firms should have lower prepayment costs due to larger loan sizes. We would also expect firms to be more aware of prepayment possibilities which would probably lead them to prepay rather early with low standard deviation.

Other candidates for subgroups could be government controlled loans with low prepayment costs and a zero tax rate or small private mortgages with very high prepayment costs.

Mean required gains and standard deviations are assumed to be constant although the maturity dependent model discussed in section 6.3 may have been more appropriate.

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[73] For MBBs issued under the cash-loan system described in section 5.3 a further complication arises because the cash-loan rate is specific to the issuing date of the individual mortgage. For some cash-loan MBB which have been open for new issues in periods with large changes in market rates it may be necessary to subgroup borrowers according to the issuing dates of their mortgages.

[74] Remaining time to maturity for the individual mortgages may vary 1-3 years according to the length of issuance period for each MBB, but this factor is probably easily accounted for by the standard deviation of required gain.

[75] Tax schemes as well as tax rates have been subject to several large changes since 1986. Corporate tax-rates on interest payments has gone from 40% to 50%, down to 38% and are now equal to 34%. A precise calculation of effective tax-rates should include the rather long credit granted to Danish firms. Taxation of households have likewise been subject to changes, but currently most households will be able to deduct interest payments at rates between 50 and 58%. Given the uncertainty of other parameters like volatility and required gain we have not found it necessary to adjust tax-rates throughout the period.

The prepayment model has been calculated for 0932027 which is an 11% annuity bond with quarterly payments issued by the mortgage credit institution 'Byggeriets Realkreditfond' (BRF) under the bond-loan system. The bond matures in year 2010 and we shall refer to it as 11-2010. Market data is taken for Wednesdays with four weeks intervals starting 25-11-87 up till 01-07-92. The initial term structure is estimated on a sample of non-callable bonds supplemented by 9% MBBs as explained in chapter 2. A constant 14% volatility of future short term interest rates have been used throughout the period.

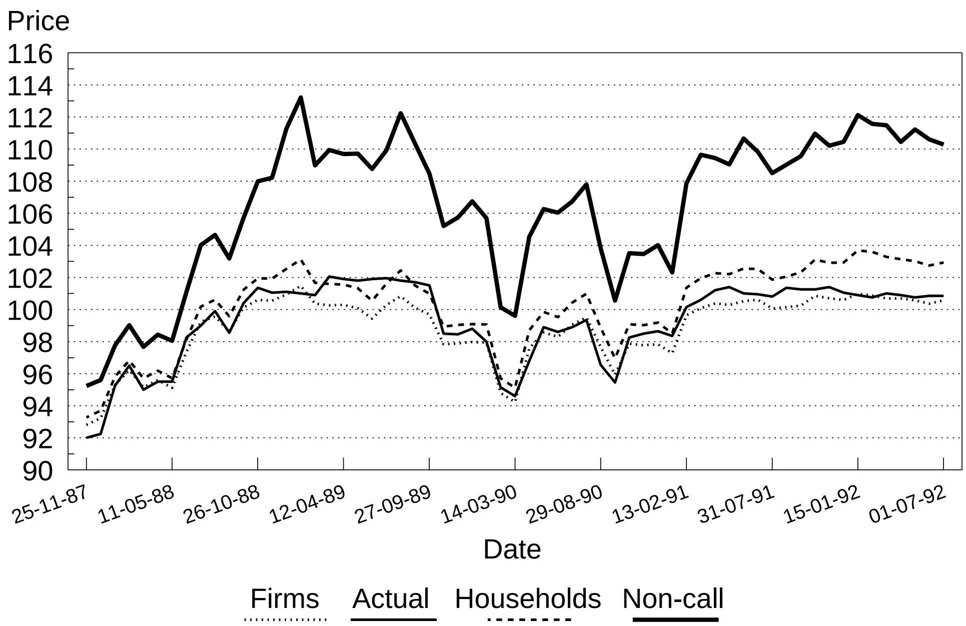

Figure 6.18: Estimated prices for firms and households compared with actual market prices and the price of a similar non-callable bond.

The actual market price for 11-2010 is shown in figure 6.18. The curve labelled 'Firms' is the theoretical price of 11-2010 if firms were the only mortgage holders in the group. 'Households' likewise refer to estimated prices from a household-only model. If the weight of households relative to firms is known the resulting theoretical price can be found as a simple weighted average. The curve 'Non-callable' refers to the estimated price of a similar non-callable bond.

Despite the casual nature of parameter estimation the model seems to fit the data remarkable well. As expected prices implied by a firms-only assumption lies below prices implied by a households-only assumption due to the lower prepayment risk of the latter. Actual market prices consistently lies between the two groups and except for the 1989-data the firm-group seems to provide the closest fit.

As a by-product of the valuation procedure the initial gain from prepayment as well as the initial prepayment rate is calculated as shown in figure 6.19. Firms have prepayment gains above households, due to their lower cost- and tax-rate. Their mean required gain is lower too which combines into far higher prepayment rates for firms.

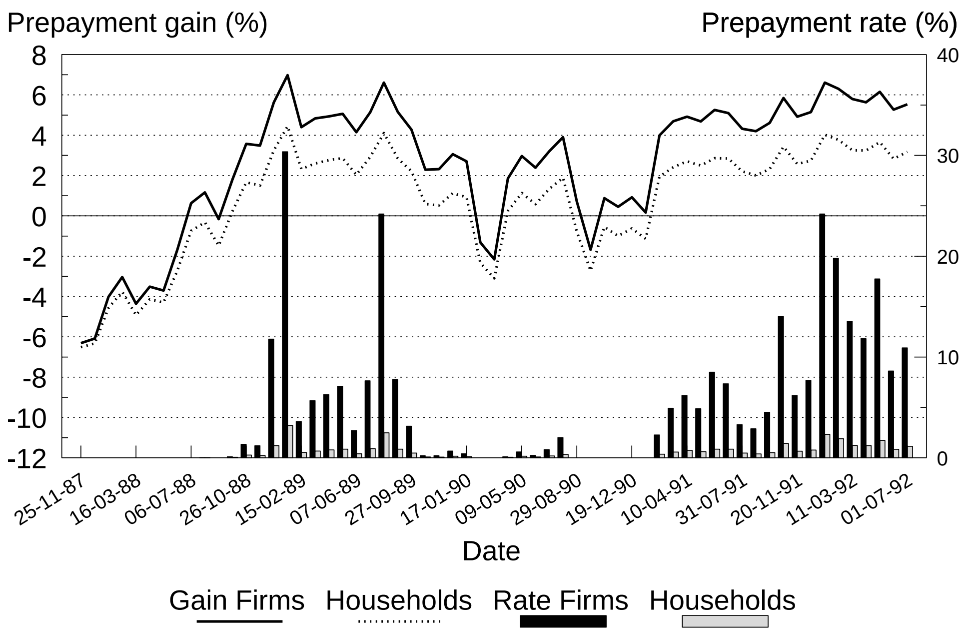

Figure 6.19: Estimated prepayment gains and prepayment rates for firms and households.

To compare with actual prepayment rates our estimates are converted into quarterly data using a two month term of notice. To give an example we find the quarterly prepayment rate for the settlement date 1-10-1991 as an arithmetic average of estimation results for the period 1-5-1991 to 1-8-1991[76]. Estimated repayment rates is given in table 6.4 together with actual repayment rates for BRF 11% 2010. Actual repayment rates on two similar 11% 2010 bond issues from other mortgage credit institutions are shown for later reference cf. the discussion below[77].

76 0.4% is added to adjust for ordinary repayment on principal. The same applies to the last four BRF-rates, which were taken from disaggregated prepayment data, supplied as a whole number.

77 Actual prepayment data has kindly been supplied by the respective mortgage institutions.

Table 6.6: *Actual and estimated repayment rates for the 11% 2010 annuity bond.*

Date	Actual BRF 0932027	Estimated Firms	Estimated House-holds	Actual KD 0923125	Actual NYK 0971677
01-Jan-88	0.25	0.40	0.40	0.25	0.38
01-Apr-88	0.35	0.40	0.40	0.34	0.34
01-Jul-88	0.36	0.40	0.40	0.33	0.35
01-Oct-88	0.35	0.40	0.40	0.34	0.41
01-Jan-89	0.52	0.77	0.49	3.58	3.97
01-Apr-89	9.97	14.85	1.95	16.89	21.72
01-Jul-89	3.70	5.58	1.06	5.39	8.42
01-Oct-89	7.95	10.81	1.56	3.40	7.17
01-Jan-90	1.02	4.11	0.87	1.88	0.37
01-Apr-90	0.44	0.83	0.52	0.31	0.44
01-Jul-90	0.44	0.43	0.42	0.54	0.42
01-Oct-90	0.46	1.32	0.59	0.42	0.45
01-Jan-91	0.47	0.40	0.40	0.46	0.45
01-Apr-91	0.56	0.40	0.40	0.46	0.63
01-Jul-91	2.40	4.86	0.96	7.35	6.09
01-Oct-91	7.40	6.40	1.10	17.01	12.06
01-Jan-92	5.40	7.53	1.18	12.70	9.51
01-Apr-92	6.40	13.09	1.66	7.88	7.29
01-Jul-92		15.45	1.85		
01-Oct-92		12.82	1.64		

Figure 6.20 compares estimation results with actual BRF-data. 1989 as well as 1991-1992 have been periods of high prepayment rates and the plot shows a striking correlation between the actual and estimated prepayment rates. Actual prepayment rates lies below firms-estimates and above households except for the settlement date 1-10-91. Like the price-result above it seems as if the bond is well described as a weighted average of firms and households.

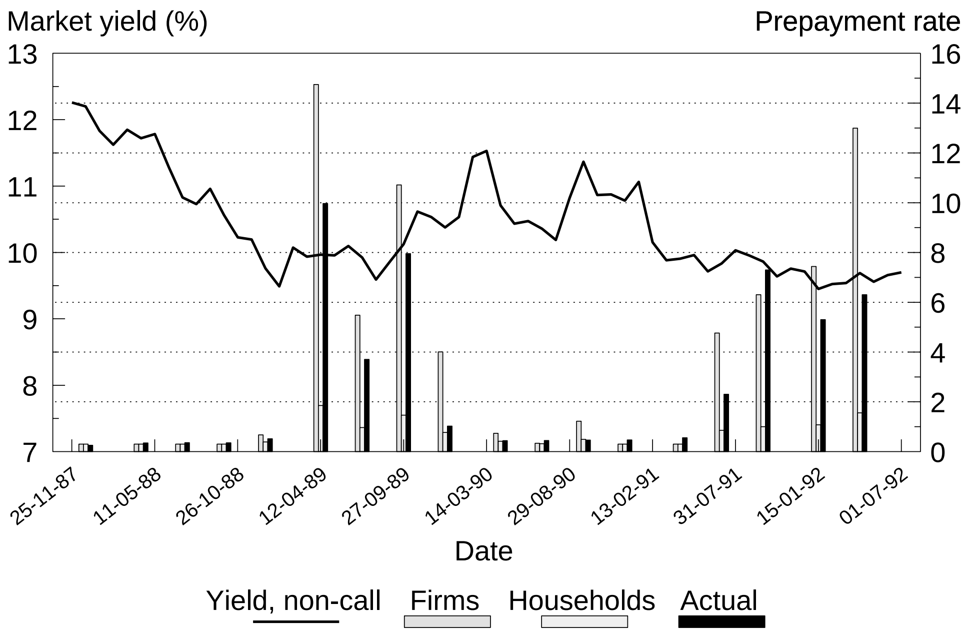

Figure 6.20: Comparison of the actual prepayment rates for BRF 11-2010 with estimated rates from a household- or firm-only model.

Table 6.7: *Data on the composition of borrowers in 0932027 BRF 11% annuity 2010.*

Date	Share of borrowers		Prepayment rate		
	Households	Firms	Households	Firms	Total
01-Apr-89	32	68	1.1	13.1	9
01-Jul-91	34	66	0.9	3.3	2
01-Oct-91	35	65	2.9	9.9	7
01-Jan-92	37	64	2.1	6.0	5
01-Apr-92	38	62	2.0	7.6	6

The mortgage institution BRF has made some data available on the composition of its borrowers[78] as shown in table 6.7. Approximately 2/3 of the entire issue consists of

78 BRF publishes prepayment data on the 24 largest 11 and 12% MBBs. The data is subdivided into corporate borrowers with loans above and below 1 million DKK and private households with mix-loan and annuity loans respectively. The data for 1-4-1989 is taken from Hasager and Møller (1989), which contains a full description of the prepayment motives for the different groups. The remaining data has been kindly delivered by BRF. BRF and Totalkredit are the only mortgage institutions which are able to provide these services.

Regrettably data are published with rather low precision and no data was obtained for the period between 1-4-89 and 1-10-91. Similar data on 9 and 10% issues are not available.

corporate borrowers. The share of firms is falling relative to housholds due to higher corporate prepayment rates.

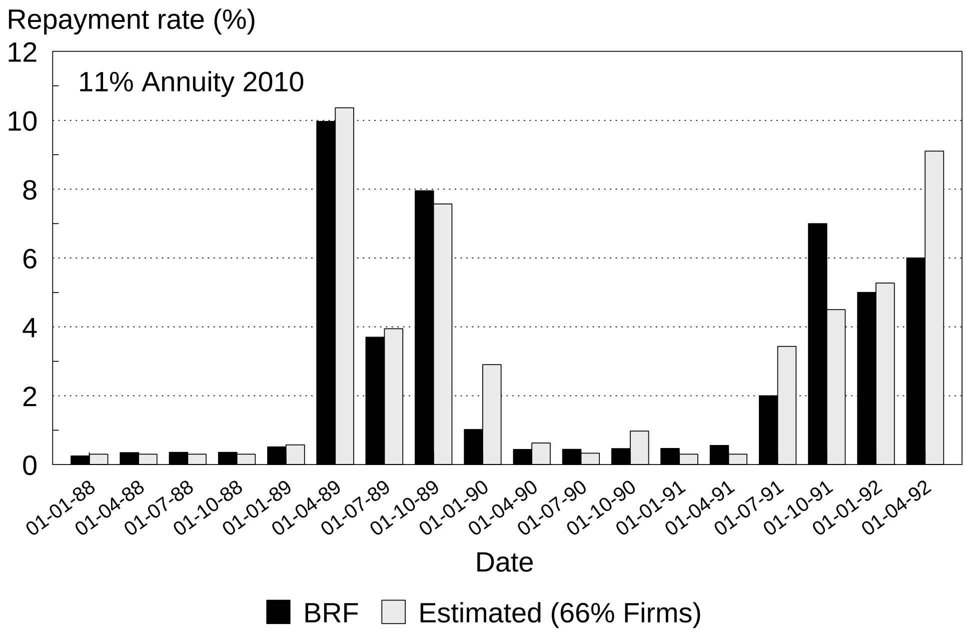

Figure 6.21: Actual versus estimated prepayment rates assuming a 66% share of firms.

Figure 6.21 compares actual prepayment rates of BRF 11% 2010 to predicted rates, using a constant 66% share of firms. The prediction of the prepayment model fits the data extremely well. The close fit is especially promising as the prepayment model uses available market data obtained between 5 and 2 month ahead of the settlement date. Investors equipped with a broader based version of the current model may not only forecast periods of high prepayment risks, they would also be given enough time to trade accordingly.

Most Danish mortgage backed bonds exist in triplicates because each of the three dominating mortgage credit institutions, Nykredit (NYK), Kreditforeningen Danmark (KD) and BRF coordinate their issues. This apply to the 11% 2010 bond as well. The three bonds have identical settlement dates, maturities, issuing periods etc. They are also priced exactly alike. On the 235 Wednesdays in the period 6-Jan-88 to 1-July-92 we found the maximum price difference to be 40 basis points. Only five observations exceeded 25 basis points and on the majority of days the difference was well below 15 basis point. The only distinguishing feature happens to be their prepayment rates as shown in table 6.4.4.

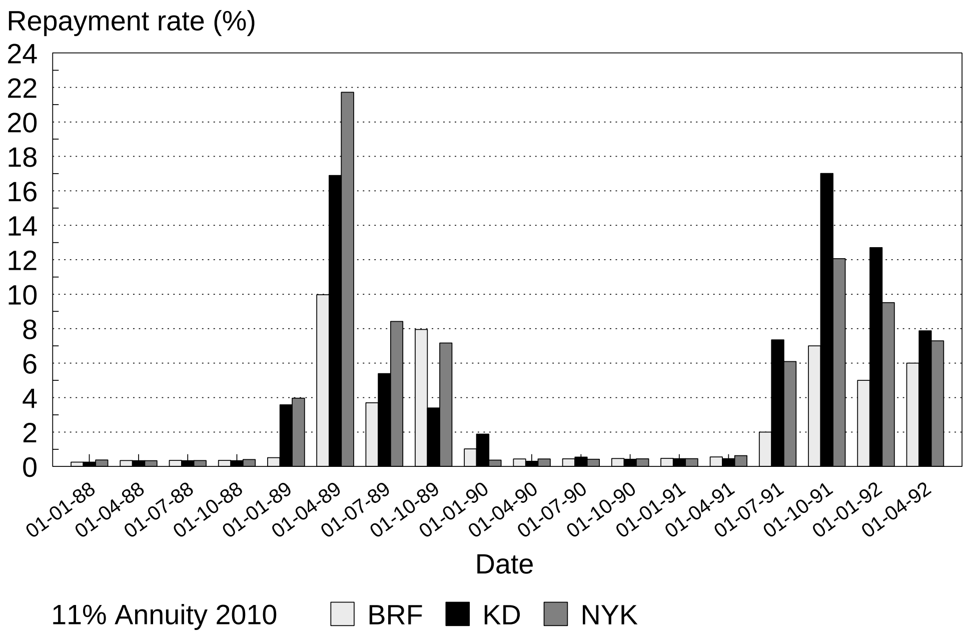

Figure 6.22: Comparison of prepayment rates for three different 11% 2010 MBBs issued by BRF, KD and NYK respectively.

Actual prepayment rates for the three different 11% 2010 MBBs are shown in figure 6.20 and it is evident that rates are higher for the NYK and KD issues. NYK and KD may have been more actively engaged in the marketing of the prepayment option, but the main reason is probably that BRF has a lower share of corporate borrowers, as discussed in Hasager and Møller (1989)[79].

Data on the composition of borrowers is not available for MBBs from KD and NYK. Taking our example beyond its limit we have therefore estimated the share of firms as the one which minimizes squared residuals between actual and predicted prepayment rates. This procedure resulted in 58% firms for the BRF-issue, 98% for KD and 108% for NYK ! Obviously this estimate exaggerates the share of firms in KD and NYK, but it nonetheless provides some evidence on possible differences accross the mortgage institutions[80].

79 Their paper predicted that MBBs issued by BRF would have lower prepayment rates than their NYK and KD counterparts. Regrettably the market impact of written information seems small.

80 A lower mean required gain for corporations would give a more realistic fit of KD and NYK prepayment data, but then the BRF-results get worse. Disaggregation of data into small and large corporate loans could probably solve this problem as we would expect KD and NYK to have a larger share of large corporate borrowing.

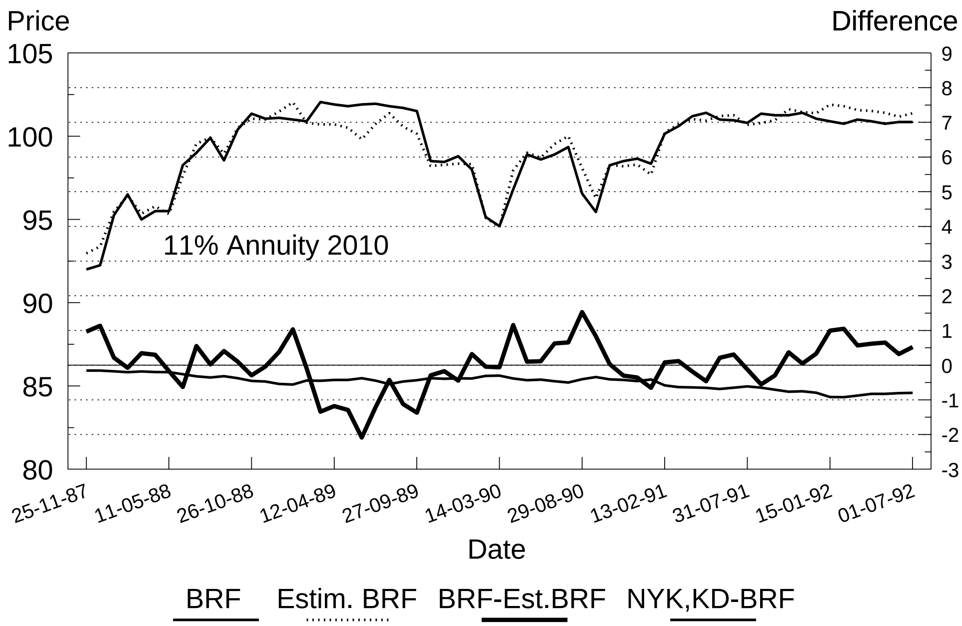

Figure 6.23:

Finally we show actual price of BRF 11-2010 (*BRF*) against the predicted price (*BRF-Estim.*) using a 66% share of firms. Note that the predicted prices are calculated with no reference to actual market prices. The prepayment model has some trouble explaining why the market keeps prices constant above 102 despite the large prepayment rates of 1989 and it somewhat overstates prices in the end of the period, but the overall fit is quite promising. The curve labeled *NYK,KD-BRF* shows the predicted difference between the BRF 11-2010 and the similar 11-2010 issued by the two larger mortgage institutions, using a 100% share of firms. This difference is not reflected in quoted market prices and from available prepayment data we therefore would expect a slightly inferior performance for the NYK and KD issues.

6.6 Conclusions

We have proposed a model with a continuous prepayment function, which directly addresses the sub-optimal nature of the prepayment decision and which is closely connected to current standards of advisory services.

The results of the model was easily explainable and they highlight the complex nature of the callable mortgage backed bonds, while still being analytically and computationally tractable. The focus on tax considerations has shown to be an important issue especially for bonds issued under the cash-loan system.

Price behaviour of mortgage backed bonds was shown to be different from the price behaviour of non-callable bonds, and any investor who uses the same methods on the two types of bonds will be in big trouble.

Clearly the model lacks data. Especially the grouping of borrowers in the individual bonds is needed. The current practice of some credit institutions of withholding these informations from the investor community while themselves actively promoting prepayment services is probably unwise, as the resulting uncertainty on MBB pricing might bias market prices downwards. This is especially a problem for the low-risk private households who participates in a mortgage pool with a large share of corporate borrowing.

For the credit institutions who provide simple grouping statistics our pricing results indicates that a different grouping especially with regard to the tax status of the mortgage owners would be appropriate. An even more relevant statistic would be the average realized gain from exercise in each group. Grouping statistics should also be published on lower coupon issues, especially the 10% MBBs.

The lack of readily avalible data on the grouping and tax-status of borrowers makes the model seems like a specialists tool. But as shown in the next chapter it will be possible to estimate the parameters of the model on available prepayment data. By parameterization of the mean required gain and standard deviation it is possible to use the same prepayment function across all MBBs. Even available data on the composition and tax-status of borrowers might be used in such a model.

7 The Estimation of a Prepayment Model for Danish MBBs

In this chapter we propose an empirical prepayment function for Danish MBBs and estimate the model on a newly constructed data-set on actual prepayments. The estimated prepayment function is incorporated into the arbitrage-free valuation framework of section 3.3 and the resulting model is used to price a range of MBBs for the period from 1988 to 1992.

It is shown that market prices are somewhat below the estimated prices for most of the period. If the model is correctly specified we would therefore expect to find a higher risk-adjusted return from investment in MBBs. An analysis of a newly constructed data-set of four-weeks holding period returns shows that MBBs have in fact performed above the level expected from non-callable bonds of similar risk. The regression technique used is of general nature and may be applied to other interesting aspects of bond return behaviour.

As a by-product of the performance analysis we find that the duration measures developed in section 6.4 have a precision, which is comparable to duration measures from non-callable bonds. This opens for a range of applications for the MBB-pricing model in the context of hedging, prediction of future returns etc. We also show that the net present value estimates are reliable predictors of future excess returns.

The prepayment data have become available very recently and due to severe time-constraints the empirical work are of a preliminary nature. We hope to follow up on these issues shortly.

The use of an empirical prepayment function is partly motivated by recent work on similar US mortage-backed securities and section 7.1 contains a review of some important papers. Section 7.2 discusses how the US-models conforms with the Danish market and argues for a somewhat different specification. Section 7.3 estimates the prepayment model while results on estimated prices and durations are shown in section 7.4. Section 7.5 contains the analysis of holding period returns. The final conclusion is given in section 7.6.

7.1 US-models for mortgage-backed securities

The valuation of US mortgage-backed securities has received increased attention in recent years. The share of securitized mortgages has rised, especially since the crisis in the Savings and Loans Associations. By securitization the risks associated with mortgage cash flows can be passed through to other investors[81].

US-mortgage backed bonds are very similar to the Danish MBBs studied in this thesis. Default risk is almost zero as payments on most MBBs are guarantied by the Government or Federal National Mortgage Associations (GNMA, FNMA) backed by the U.S. government. Each MBB consists of a large pool of mortgages in which each individual borrower may prepay the loan at any time at small prepayment costs. Mortgages are given as annuities. Judging from the papers some differences remain. US mortgages have monthly payments against semiannual or quarterly for the Danish market. US mortgages often have due-on-sale clauses while Danish mortages are assumable. The largest part of US MBBs are Single-Family pools, restricted to residential properties[82], while a Danish pool may contain small and medium sized residential mortgages together with very large corporate mortgages. Finally an active US market exists for adjustable rate MBBs as well.

This section discusses some models proposed for US mortgage backed bonds. The US models typically consists of a prepayment model and a term structure model. The prepayment model explains historical prepayment rates by interest rates and a set of explanatory variables. The stochastic model of the term structure is estimated separately on non-callable bond data. Finally MBB-prices are found by a Monte-Carlo simulation in which the term structure model generates paths of future interest rates and the prepayment model generates the corresponding interest rate dependent cash flows.

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81 As of 1989 about one-third of residential US-mortgages have been securitized, cf. Breeden(1991,p.85). The corresponding number for the Danish market is larger with 63% of all mortgages being issued as MBBs. Numbers are taken from The Quarterly Bulletin of the National Bank of Denmark(November 1991,p.23) as an average of mortgages held by mortgage credit institutions relative to all new mortgages from 1981 to mid 1991.

82 Cf. Schwartz and Tourus(1989).

As seen below the distinction between the prepayment and the term structure model is not a clear-cut one as the choice of term structure model determines the types of interest rates available for the explanation of prepayments and vice versa.

7.1.1 Common factors explaining prepayments.

The following factors are often[83] listed as the main explanations of conditional prepayment rates (CPR)[84].

- The *refinancing incentive* is equivalent to the gain from prepayment studied in chapters 5 and 6. Borrowers tend to prepay high coupon loans, when market rates are low. This is an important factor even for adjustable rate MBBs as shown by McConnell and Singh(1991).

- *Premium Burnout* - This effect is caused by a changed composition of borrowers. A premium mortgage pool[85], which has experienced high CPR for a while, tend to get lower CPR as time passes, because the mortgagors most inclined to prepay depart from the pool.

- *Households mobility* cause prepayments because most US mortgages are due-on-sale. Household mobility is seldom used directly. Instead it is captured by the following factors:

- *Mortgage age*. Prepayments on par mortgages start at 0%, but as people start to move, CPR increases, reaching a fully *seasoned* annualized level at about 6% after two years[86]. *Discount pools* may season slower, as households are deterred from moving. The long run influence from mortgage age seems to be an open question. Lifecycle considerations would lead to increased prepayments as mortgage owners age, but some authors observe the opposite trend.

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[83] Cf. Richard and Roll(1989).

[84] Monthly CPR is measured as \$prepayed divided by (\$balance$_{t-1}$ - \$scheduled principal paid)

[85] In Richard and Roll(1979) the terms discount, par and premium pools refer to pools in which the coupon rate is below, at par with or above current refinancing rates.

[86] Richard and Roll(1989).

- *Month of the year* - This refers to a distinct seasonal pattern with low CPR in February-March and high CPR in October-November. With a lag due to the term of notice this can once again be explained by households mobility.

- *Macroeconomic factors:* Breeden (1991), using data back to 1979, shows that during the recession period 1980-1982 prepayment rates were low compared to the following period of consistent growth. Again household mobility is the main explanatory variable.

- *Geographical variation* may exist due to differences in mobility.

The papers reviewed below generally agree on the explanatory variables, but their practical implementations differs.

7.1.2 Green and Shoven (1986)

The paper by Green and Shoven (1986) studies the impact of interest rates on prepayment rates motivated by the liquidity problems of US Savings and Loan Associations. No formal valuation model is given. The paper uses individual data on 4000 mortgages held by two large Californian Savings and Loans Associations. Each mortgage is observed from 1975 to 1982. Known characteristics for each mortgage includes original mortgage amount, original length of mortgage, coupon rate, starting year, and date of prepayment if prepayed.

The prepayment function is estimated by the proportional hazard model proposed by Cox(1972)[87]. In the present context the hazard rate $\lambda(t, x)$ is the continuous rate of prepayment at time *t* given that the mortgage has not been prepayed prior to time *t*. $\lambda(t, x)$ is simply the continuous time equivalent to the prepayment function used in chapter 5 and 6. *x* is a vector of explanatory variables. The proportional hazard model can be formulated as

(7.1)
$$\lambda(t, x) = \lambda_0(t) \exp\left(\sum_{i=1}^{n} \beta_i \cdot x_i \right)$$

[87] Kiefer(1988) contains an excellent survey of hazard models.

where the prepayment rate is factored into the so-called *baseline hazard*, $\lambda_0(t)$, which depends on mortgage age t only, and a term which depends on the parameter vector x. GS use an exponential formulation with β representing a vector of unknown coefficients to be estimated.

The empirical specification uses only one explanatory variable in that x equals the *lock-in* variable defined by

$$\text{lock-in} = \frac{\text{face value - market value}}{\text{initial principal amount} \times I_t} \tag{7.2}$$

I_t is the general price increase on houses since start of mortgage. Lock-in can be seen as a measure of the disincentive to prepay and as expected GS find a negative association between lock-in and prepayment rates. The model was estimated by partial likelihood[88] with due account of censoring.

The estimated baseline-hazard measures the rate by which a t-year mortgage prepay, provided the lock-in variable is zero. Baseline prepayment rates are largest between 3-7 years and after 14 years, confirming the mortgage age effect discussed above[89].

7.1.3 Schwartz and Tourus (1989)

Schwartz and Tourus(1989) describe a complete valuation model for MBBs. The prepayment function is estimated by a proportional hazard approach but contrary to Green and Shoven(1986), ST have no data on individual mortgages. Instead they construct a pseudo-individual data-set from monthly CPR for 27 large 30 year GNMA Single-Family pools observed in the period from January 1978. By assuming a common principal of $100,000, ST are able to calculate the number of mortgages originally issued, the number prepaying at time 1, 2, 3 etc. for each of the MBBs, and the number censored, i.e. the mortgages still active when the sample period ends November 1987.

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[88] Cf. Kalbfleisch and Prentice(1980,ch. 4). Partial likelihood allows β to be estimated without a specification of $\lambda_0(t)$. Statistical software like LIMDEP or SAS includes routines for the estimation of most proportional hazard models.

[89] In a related study Quigley(1987) the proportional hazard model is used to explain data on household mobility. His estimations includes a large range of socio-economic factors, but a lock-in measure similar to Green and Shoven(1986) is shown to reduce household mobility and thereby indirectly prepayments.

Schwartz and Tourus apply a proportional hazard model with the baseline hazard parameterized by a log-logistic function and four explanatory variables. The model is estimated by maximum likelihood.

The refinancing incentive is modelled as a simple difference between coupon rate and refinancing rate $v_1(t) = c - l(t-s), \quad s \geq 0$ were t denotes mortgage age at prepayment or censoring, c is the coupon rate and $l(t-s)$ the long-term Treasury rate lagged s periods. To account for non-linearities in the refinancing relation, ST use the cube of $v_1(t)$, as a second explanatory variable. Premium burnout is modelled by $v_3(t) = \ln(AO_t / AO_t^*)$. AO_t is the dollar amount of the pool outstanding at time t, while AO_t^* is the dollar amount that would prevail at time t in the absence of prepayments. A low value of v_3 should therefore indicate a burned out pool consisting mainly of slow prepayers. Finally ST employ a seasonal dummy, $v_4(t)$ equal to one between May and August and zero otherwise.

The empirical results show that prepayments increase more than linearly with the difference between coupon rate and the refinancing rate lagged three months. As expected burned-out pools have lower prepayment rates, but the seasonal dummy turns out to be insignificant[90]. The baseline prepayment rate increases up to a maximum at 6 years and diminishes thereafter with mortgage age.

The estimation of the prepayment function was only the first step. Turning to the term structure models ST specify and estimate a version of the two-factor model proposed by Brennan and Schwartz(1979). This model explains the term structure by the correlated evolution of the short term rate and the rate of a long term console bond. The choice of this two-factor model corresponds with the prepayment specification, which needs a long term rate to model the refinancing incentive. As noted in section 3.1 this ’traditional’ type model will not fit the current term structure, but ST employ a sort of calibration procedure by determining a value of the market price of short term risk, which allow the model to price a specific non-callable mortgage correctly.

Finally the prepayment function is integrated into the term structure model. As the refinancing incentive as well as the burn-out effect depends on the history of interest rates the model is path-dependent, and ST solve it by use of a Monte-Carlo simulation procedure. Results given for some initial values of short and long term rates are

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[90] Comparing with Richard and Roll(1989), the seasonal dummy should probably have been set to 1 in August-November instead, due to the prepayment lag.

broadly equivalent to the price curves shown in section 6.3 except for the fact that the high baseline prepayments explained by household mobility allow the MBB to exceed the value of a comparable non-callable bond for high values of the long rate. The initial spread between short and long rates has a surprisingly small effect on long term bond prices, indicating that the term structure converges very rapidly toward a constant long rate level.

7.1.4 McConnell and Singh(1991)

Schwartz and Tourus(1989) may seem complex, but the paper by McConnell and Singh(1991) contains perhaps the most elaborate valuation model ever seen. They set out to value adjustable rate mortgage-backed bonds (ARMBB). These securities are similar to fixed coupon MBBs, except that coupon rates are annually adjusted to the one-year Treasury bond yield index. The ARMBBs have annual adjustment caps of ±2% as well as a life-time cap typically 6% above initial coupon rates. Borrowers are allowed to prepay at any time.

The MS-model is quite similar to Schwartz and Tourus(1989). Data consists of conditional prepayment rates, which are converted to mortgage duration data by the $100,000-trick described above. MS use a proportional hazard model with seven explanatory variables to estimate the prepayment function. The refinancing incentive is measured by recent changes in short rates. The burned-out effect is captured by a dummy, which equals one, if the prevailing short rate is below the minimum for the last 12 months. The idea is that only a rate lower than previous rates will induce some of the remaining mortgage holders to prepay. Seasonality is included as well as variables connected to cap-features and yield curve slope. Yield curve slope measures prepayment incentives by switchers, who change to a fixed-coupon mortgage.

A two factor model is estimated as in Schwartz and Tourus(1989), but calibrated to match the price of a non-callable ARM-backed bond. Integration of the prepayment function is more complicated due to several new state variables measuring minimum rates, capped coupons etc. Examples of ARM-backed sequrity prices are finally obtained by Monte-Carlo simulation. Regrettably no information is given on computational aspects.

The simulation results by McConnell and Singh show that prices of ARM-backed bonds are quite volatile in the absence of prepayments due to the adjustment lag and cap-features. Prepayments dampen volatility considerably as the MBB-values are driven towards par.

7.1.5 Jacob, Lord and Tilley(1987)

Jacob, Lord and Tilley(1987) provide an excellent introduction to the pricing of MBBs, but their paper lack some detail. JLT state the explanatory factors given above and give the following simple example of a prepayment function

(7.3)
$$\ln \text{CPR} = a_0 + a_1 K + a_2 \sum_{j=1}^{4} (c - r_{T-j})$$

where CPR denotes monthly conditional prepayment rates, K is an age- and month-of-year dependent function, c denotes the coupon rate and r_{T-j} is the 10-year Treasure yield, lagged j months. Taking exponentials the model resembles the proportional hazard specification given above[91]. The last term measures the refinancing incentive as the average difference between coupon rate and a lagged 10-year Treasury yield. The actual model used by Morgan Stanley, New York, contains a more sophisticated specification.

The term-structure model is a single factor arbitrage-free BDT-model. Judging from the paper, JTL first use the model to find 10 year Treasury yields along a sample of paths, probably by repeated use of a backwards pricing procedure and then the standard path-dependent valuation procedure is used, with MBB cash flows given by the prepayment function. A few comforting pricing results are shown.

7.1.6 Richard and Roll(1989)

Richard and Roll(1989) give an extensive discussion of prepayment motives and present the prepayment model developed at Goldman Sachs, New York. Their model for GNMA Single-Family pools are based on 103,694 observations on monthly CPR for the period May 1979 to May 1988. RR explain CPR as the product of four effects:

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[91] No information is given on estimation techniques, but it looks as if a direct regression approach is used.

(7.4) CPR =(Refinancing Incentive) x (Seasoning Multiplier) x (Month Multiplier) x (Burnout Multiplier)

Each term in the multiplicative relationship is separately modelled and the combined model estimated by weighted least squares. MBBs receive weights according to their nominal balances and old observations receive less weight than newer observations.

RR argue that the basic economic variable measuring the borrowers refinancing incentive is A/P, that is present value, A, of the annuity, divided by outstanding principal P. The coupon rate, C, divided by the refinancing rate, R, provides a reasonable approximation, at least for long mortgages, and C/R is used throughout their model. The refinancing incentive is estimated as a function of C/R starting at 4% for discount pools (C/R=0.6), rising slowly to 8% for C/R=1.1 and then sharply increasing with 20% for C/R=1.2 and 45% for high premium pools (C/R=1.4). The seasoning multiplier gives the fraction of long-run prepayment level reached as a function of mortgage age. Once again this depends on C/R with premium pools reaching 100% of long-run level in two years, while discount pools takes nearly ten years. The month-of-year effect is estimated as a set of multipliers, with March prepayment 25% below and November prepayment 25% above average level.

The premium burnout effect is explained by differences in prepayment costs among individual borrowers. This causes prepayment rates to be substantially lower the second time the pools reaches a certain low level of interest rates. Some prepayments may occur however as prepayment costs change over time. At Goldman Sachs the effect of premium burnout is captured through a "complicated non-linear function. This function depends on the entire history of C/R since the mortgage was issued." (p.76).

The authors show some impressive results on their models in-sample predictions of average prepayment rates for different coupon rates and refinancing ranges. The overall R-squared for monthly CPR-predictions, when MBBs are grouped by coupon and remaining time to maturity, is 94.6%

No valuation model is presented, but a Monte-Carlo simulation is shown in which the refinancing rate R follows a random walk with 8% annual variance. The path of CPR clearly shows the effect of premium burnout.

7.2 Modelling Danish prepayment data

The previous section surveyed papers published on the valuation of US mortgage-backed securities. Only three of the papers described a combined prepayment and valuation model, but similar models are probably in use as proprietary models in investment firms etc.

Despite their complexity all US-models agree on the central prepayment motives as well as the path-dependent nature of MBB-valuation. Before trying to implement such a model for the Danish market one should note some important differences.

All Danish mortgages are assumable and a priory household mobility should therefore play a minor role in the explanation of prepayment. We would not expect to find significant seasonal effects and baseline prepayment rates should be lower[92]. Geographical differences can hardly play a role in the Danish market.

The loans issued during the cash-loan system described in section 5.3 are still a major part of outstanding mortgages and these loans have lock-in effects due to taxation. Our model should incorporate some adjustment for these loans.

Danish 20-year mortgage pools consist of corporate as well as household mortgages. Corporate loans are much larger, and as indicated in section 6.5 their prepayment rates are higher. A significant burnout effect would thus be expected for these MBBs as corporate borrowers leave the pools. On the other hand, we do not know the exact share of corporate loans, and high prepayment rates could therefore signal a large share of firms which would indicate higher future prepayment rates.

Summarizing we expect that a model for Danish prepayment data should include the prepayment incentive and the burnout effect. MBBs with corporate borrowing as well as MBBs issued under the cash-loan system require separate treatment.

Path-dependency seems to be an invariable characteristic of MBB valuation models. The path-dependency stems either from the lag between the prepayment decision and the actual prepayments or from the burn-out effect.

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92 The change in legislation May 1992, which allow borrowers to prepay and take a new 30-year mortgage may lead to high prepayment rates for mortgages with few years remaining, cf. section 6.2. This effect is not covered in the data available.

In the current analysis we shall nevertheless use a path-independent MBB-pricing model for several reasons.

The path-independent specification is simpler to implement and generally far more efficient. To employ a path-dependent specification required a complete rewrite and test of our valuation model, which was not possible within the time-limits of this thesis. Monte-Carlo techniques furthermore introduces sampling errors. Finally we wanted a PC-model, able to compute prices, modified durations and convexities for hundreds of MBBs several times during a single trading day. The papers reviewed above present no information on calculation times for their models, but judging from Cheyette(1992) a well designed standard Monte-Carlo approach might easily run 1-2 minutes for a single valuation, while the path-independent model used below needs 4-6 seconds on a comparable PC[93].

Secondly a backward pricing path-independent procedure allows for a precise modelling of the refinancing incentive. As noted by Richard and Roll(1989) the borrower possesses an American option and his refinancing decision should be based on the present value of the mortgage relative to remaining principal. This is exactly the approach taken in this thesis. But while present values and option hold-on values are easily calculated at each date-event for a backwards pricing procedure, they are not readily available, when a single interest path is followed from time zero to maturity. This is probably why the path-dependent models reviewed above all use approximate measures like differences or ratios between coupon and refinancing rates. These specifications does not fully capture the assumed stochastic evolution of the term structure as well as the change in mortgage volatility as time to maturity decreases. The complicated interactions between taxation and time to maturity, documented in section 5.3, would also be hard to implement for a path-dependent Monte-Carlo procedure.

Thirdly our model already includes the prepayment lag caused by the term of notice. As described in section 3.3 the prepayment function is specified at the decision date and based on the present value of a mortgage prepayed at the later settlement date. In a path-dependent specification the prepayment function is specified at the settlement date depending on lagged interest rates. No significant difference exists between these two approaches.

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93 Cheyette(1992) presents a ’representative path method’, which produces path-dependent valuations significantly faster than the standard Monte-Carlo approach, but the exact algorithm is not given.

Finally the burnout effect may be described as a case of unobserved heterogeneity. Consider a simple example, in which the pool is divided into firms with low prepayment costs and households with high prepayment costs. First time the interest rate hits a low level r^* most firms prepay, but only a few households. Next time the level r^* is reached fewer prepayments occur, as firms have left the pool. Aggregate prepayment data would show clear signs of a path-dependent burnout effect.

Path-dependency arises because the proportion of firms relative to households changes troughout the path. If the proportion of firms could be observed at time zero, the need for a path-dependent specification disappears. The MBB should be viewed as a portfolio of two pools consisting of firms and households respectively. Each pool could be valued separately by a path-independent valuation model and the final MBB-value as a simple weighted average. More or less directly this is the approach suggested by all papers on Danish MBB-valuation.

As noted in section 6.5 at least one Danish mortgage association makes limited data on the composition of borrowers available. But even without such data one might at least in principle estimate the composition from historical prepayment data. Estimation techniques which accounts for such unobserved heterogeneity have been applied in the context of labour market models, cf. Kiefer(1988). The complicated burnout effects of the US models captures the change in borrowers composition indirectly, by observing how historical prepayment rates have changed along the observed path of interest rates.

The conclusion is that the path-independent model of section 3.3 may be quite suitable for the valuation of MBBs. The model is easy to implement, computationally efficient and void of sampling error. In some respects it allows for a more realistic description of the reinvestment incentive, and lags between the borrowers decision and the final prepayment can be modelled at little extra costs. The burnout effect can be reinterpreted as a case of unobserved heterogeneity and valuation results similar to the path-dependent model may be obtained by calculations conditioned on an initial distribution of borrowers[94].

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[94] The path-independent model is not generally applicable. The way in which caps change the cash-flow of ARMBBs in the model of McConnel and Singh(1991) requires the use of a path-dependent model. The same goes if borrowers are viewed as technical traders, who base their reinvestment decision on past changes in interest rates. New techniques, cf. Cheyette(1992), might hopefully produce much faster path-dependent valuation algorithms.

7.3 Data and estimation

This section reports estimation results from several versions of a prepayment model for Danish bond data. All estimations are done using observed quarterly prepayment rates for 69 Danish bonds in the period 1988 to 1992. The estimation are done in a probit model framework. As shown below this specification follows directly from our model of the individual borrowers refinancing decision. The next section discusses pricing results from one of the estimated prepayment models.

To obtain the data on observed prepayment rates for Danish MBBs, we made contact to the three dominating mortgage credit associations, 'Kreditforeningen Danmark' (KD), 'Nykredit' (NYK) and 'Byggeriets Realkredit Fond' (BRF). All associations kindly supplied copies of their published drawing lists for the period 1985-1992. Computerized data was regrettably not available.

The data was transformed into computer readable form by the use of a scanner in combination with optical character recognition software (OCR)[95]. Although OCR beats manual entry it is still a very time consuming process and the format and quality of the printed material have been a limiting factor. Some lists did not contain the necessary bondcode information while poor printing quality made the BRF-lists illegible to the OCR-software. These data have therefore been omitted in the present analysis.

The basic data-set consists of 11704 individual repayment observations. For this preliminary study a subset of 1032 observations was selected coming from 69 MBBs with nominal outstanding amount above 500 million DKK. To increase homogeneity all bonds were required to have quarterly payments. This condition excludes bonds issued under the cash-loan system cf. section 5.3. We finally confined ourselves to the period from April 1988 to July 1992, which as seen in section 2.3 includes the two major periods of low interest rates. Table 7.1 shows a list of all bonds with some statistics.

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[95] The author is indebted to cand.merc. Henrik Sørensen, GTJ FinansAnalyse, who has prepared the prepayment data for the current analysis.

Table 7.1: *Summary of prepayment data for 69 large MBBs with quarterly payments in the period from April 88 to July 92.*

BondId	Credit Asso-ciation	Coupon rate	Name	Last Year	Bond Type	High Risk	Average prepay-ment rate (%)	Maximum prepay-ment rate (%)	Burnout rate, July 1992
0971804	NYK	12	2.S.	2007	ANN	0	6.93	34.59	0.254
0922676	KD	12	22.S	2007	ANN	0	6.41	29.57	0.292
0923184	KD	12	22.S.	2010	ANN	1	11.04	37.21	0.110
0971790	NYK	12	2.S.	2010	ANN	1	11.85	47.67	0.085
0922684	KD	12	23.S	2017	ANN	0	4.97	15.53	0.477
0971782	NYK	12	3.S	2017	ANN	0	5.26	22.84	0.382
0971774	NYK	12	3.S.	2020	ANN	1	9.50	36.72	0.159
0923176	KD	12	23.S.	2020	ANN	1	6.83	17.56	0.293
0922323	KD	11	22.S.	2007	ANN	0	2.70	9.79	0.637
0971421	NYK	11	2.S.	2007	ANN	0	2.81	15.87	0.634
0971650	NYK	11	12.S.S	2010	SER	0	0.84	2.50	0.875
0923567	KD	11	52.S.S.	2010	SER	0	0.92	2.98	0.865
0971677	NYK	11	2.S	2010	ANN	1	4.37	21.35	0.458
0923125	KD	11	22.S.	2010	ANN	1	4.24	16.53	0.462
0924512	KD	11	22A.S.	2012	ANN	1+	23.37	61.49	0.259
0973297	NYK	11	2A.S.	2012	ANN	1+	17.07	59.59	0.208
0971413	NYK	11	3.S.	2017	ANN	0	1.79	12.24	0.736
0922315	KD	11	23.S.	2017	ANN	0	2.04	9.54	0.706
0923117	KD	11	23.S.	2020	ANN	1	2.58	9.49	0.643
0971669	NYK	11	3.S	2020	ANN	1	2.30	9.93	0.680
0921157	KD	10	22.S.	2006	ANN	0	0.41	1.86	0.941
0971316	NYK	10	2.S.	2007	ANN	0	0.26	1.48	0.962
0922463	KD	10	52.S.S	2007	SER	0	0.50	2.04	0.930
0971480	NYK	10	12.S.S	2007	SER	0	0.50	2.23	0.929
0922188	KD	10	22.S.	2007	ANN	0	0.33	1.81	0.956
0971693	NYK	10	12.S.S	2010	SER	0	0.42	1.43	0.936
0923079	KD	10	22.S.	2010	ANN	0	0.20	1.64	0.971
0923524	KD	10	52.S.S.	2010	SER	0	0.40	1.50	0.944
0971715	NYK	10	2.S.	2010	ANN	0	0.21	1.05	0.971
0973262	NYK	10	12A.S.S	2012	SER	0	0.32	0.72	0.981
0924504	KD	10	22A.S.	2012	ANN	0	0.00	0.13	1.000
0973254	NYK	10	2A.S.	2012	ANN	0	0.16	0.33	0.991
0970271	NYK	10	3.S.	2017	ANN	0	0.13	0.81	0.983
0921165	KD	10	23.S.	2017	ANN	0	0.19	1.46	0.970
0923532	KD	10	53.S.S.	2020	SER	0	0.12	0.83	0.983
0971685	NYK	10	13.S.S	2020	SER	0	0.17	0.71	0.977
0971707	NYK	10	3.S.	2020	ANN	0	0.12	0.62	0.984
0923060	KD	10	23.S.	2020	ANN	0	0.12	1.27	0.983
0973270	NYK	10	3A.S.	2022	ANN	0	0.11	0.41	0.994

0924644	KD	10	53A.S.S	2022	SER	0	0.04	0.29	1.000
0924539	KD	10	23A.S.	2022	ANN	0	0.10	0.34	0.995
0973289	NYK	10	13A.S.S.	2022	SER	0	0.14	0.24	0.993
0921181	KD	9	21.S.	1996	ANN	0	0.31	1.43	0.946
0924156	KD	9	21.S.	2000	ANN	0	0.07	0.67	0.992
0921203	KD	9	22.S.	2006	ANN	0	0.00	0.03	1.000
0970433	NYK	9	2.S.	2006	ANN	0	0.00	0.03	0.999
0921912	KD	9	22.S.	2007	ANN	0	0.02	0.23	0.999
0970522	NYK	9	2.S.	2007	ANN	0	0.01	0.16	0.999
0971979	NYK	9	12.S.S.	2010	SER	0	0.16	0.99	0.976
0924172	KD	9	22.S.	2010	ANN	0	0.01	0.33	0.997
0971936	NYK	9	2.S.	2010	ANN	0	0.04	0.38	0.993
0924229	KD	9	52.S.S	2010	SER	0	0.03	0.39	0.995
0924490	KD	9	22A.S.	2012	ANN	0	0.00	0.08	1.000
0973203	NYK	9	2A.S.	2012	ANN	0	0.06	0.14	0.996
0924598	KD	9	52A.S.S	2012	SER	0	0.01	0.12	1.000
0973211	NYK	9	12A.S.S.	2012	SER	0	0.16	0.28	0.990
0970441	NYK	9	3.S.	2017	ANN	0	0.01	0.05	1.000
0921211	KD	9	23.S.	2017	ANN	0	0.00	0.01	1.000
0924237	KD	9	53.S.S.	2020	SER	0	0.03	0.37	0.995
0971987	NYK	9	13.S.S.	2020	SER	0	0.02	0.15	0.997
0971944	NYK	9	3.S.	2020	ANN	0	0.03	0.21	0.996
0924180	KD	9	23.S.	2020	ANN	0	0.00	0.06	0.999
0924636	KD	9	53A.S.S.	2022	SER	0	0.00	0.06	1.000
0924520	KD	9	23A.S.	2022	ANN	0	0.00	0.02	1.000
0973246	NYK	9	13A.S.S.	2022	SER	0	0.09	0.16	0.994
0973238	NYK	9	3A.S.	2022	ANN	0	0.03	0.08	0.998
0970859	NYK	8	2.S.	2006	ANN	0	0.01	0.09	0.999
0921637	KD	8	22.S.	2006	ANN	0	0.03	0.18	0.995
0921645	KD	8	23.S.	2017	ANN	0	0.00	0.01	1.000

For each settlement date t and bond m the published drawing lists supply total repayment rates, TRR_{mt}, i.e. the sum of scheduled and non-scheduled repayments as a fraction of outstanding nominal value. Scheduled repayment rates, SRR_{mt}, have been calculated based on the normal amortization schedule and the prepayment rate, λ_{mt}, was finally found as $\lambda_{mt} = (TRR_{mt} - SRR_{mt})/(1 - SRR_{mt})$. Table 7.1 shows average and maximum prepayment rates for each bond in the sample. The overall maximum prepayment rates come from two 11% 2012 bonds with rates around 60%. Low-coupon bonds show almost zero prepayments which confirms the conjecture of section 7.2 that housing turn-overs play an almost negligible role on Danish data compared to US prepayment experience. Figure 7.1 shows average prepayment rates for different coupon rates and settlement dates.

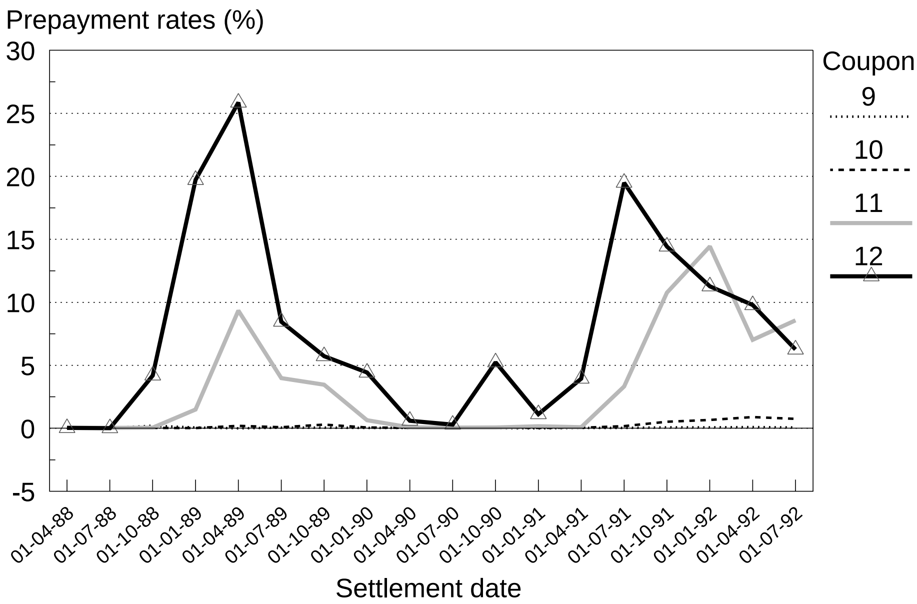


Figure 7.1: Average prepayment rates for different settlement dates and coupon rates.

The required gain model of chapter 6 lends itself directly to a statistical formulation. Consider a MBB m in which each borrower i has a required gain g_{mti} at time t. The required gain is assumed to be normally distributed across borrowers with mean required gain of g^*_{mt} and standard deviation σ_{mt}. At time t borrowers observe the *actual gain from prepayment*, g_{mt}. Let the decision variable y_{mti} equal one, if borrower i prepays at time t and $y_{mti} = 0$ otherwise. The probability that a single individual prepays is thus given by

$$\text{Prob}[y_{mti} = 1] = \text{Prob}[g_{mti} < g_{mt}] \tag{7.5}$$
$$= \Phi\left(\frac{g_{mt} - g^*_{mt}}{\sigma_{mt}}\right) = \pi_{mt}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function. The observed fraction of prepayed loans[96], λ_{mt}, from bond m at time t will thus be an estimate of the true prepayment probability π_{mt}.

96 We implicitly assume all loans in a single pool to be of equal size.

The model is a special case of the so-called *probit model* often used to model economic decision problems[97]. A more general specification of a probit model assumes that the decision variable y_{mti} depends on a vector of explanatory variables, $x_{mt} = \{x_{mtk}\}$, $k = 1, ..K$

$$\text{Prob}[y_{mti} = 1] = \Phi\left(\sum_{k=1}^{K} \beta_k x_{mtk}\right) \tag{7.6}$$

where $\beta = \{\beta_k\}$, $k = 1, .., K$ is a vector of unknown coefficients. Note that the explanatory variables may depend on m and t, while β is constant.

The unknown parameter vector β can be estimated by maximum likelihood given observed quarterly prepayment rates, λ_{mt}, as well as a vector of explanatory variables x_{mt}. The log likelihood is given by

$$\ln L = \sum_m \sum_t \lambda_{mt} \ln \Phi(\beta' x_{mt}) + (1 - \lambda_{mt}) \ln(1 - \Phi(\beta' x_{mt})) \quad . \tag{7.7}$$

The log likelihood is globally concave, cf. Amemiya(1981,p.1495), provided all prepayment rates lie between zero and one and the maximization can be done by standard Newton-Raphson techniques[98].

The choice of explanatory variables reflects the choice of pricing model. We are mainly interested in variables which are either constant or easily calculated as part of the backwards pricing procedure, cf. the discussion of the previous section.

The pre-tax gain from prepayment, g_{mt} defined by $g_{mt} = (B_{mt} - W_{mt})/B_{mt}$ is used as the primary indicator of the refinancing incentive. B_{mt} denotes the present value of scheduled payments from the bond discounted by the term structure of non-callable bonds[99]

---

[97] Probit-models may be estimated on individual decision data as well. For a survey of probit and related models see Greene(1990, ch.20-21) or Amemiya(1981).

[98] We have used the statistical software system LIMDEP ver. 5.1 developed by prof. William H. Greene. The sample contains some observations with $\lambda_{mt} = 0$. These have been substituted by a small number (0.03%) as suggested in Greene(1989,p.203).

[99] We have used the non-callable flat-spline model discussed in section 2.3. This model was judged to be the best model for the 1988-92 period.

while W_{mt} denotes the prepayment value, i.e. the present value of the remaining payments provided the borrower decides to prepay his loan at time *t*. W_{mt} corresponds to remaining principal adjusted for prepayment costs of 2% and a 65 days term of notice[100].

The gain from prepayment was calculated with four weeks intervals for all 69 bonds in our sample for the period from January 1988 to June 1992. To get a unique measure of the gain corresponding to each quarterly settlement date we have chosen the maximum gain obtained in the three month period ending 65 days before each settlement date[101]. This measure is labelled *MGAIN00* in the tables below.

In general the gain from prepayment obtained from this procedure will differ from the actual gain presented to the mortgage owner. The mortgage owner will typically prefer to refinance in a new lower-coupon MBB, perhaps a 9% bond. As shown in section 2.3 these bonds have had higher yields than corresponding non-callable bonds. This may overstate the refinancing incentive especially for the last part of the sample period.

A second objection could be that borrowers use an after-tax valuation procedure as discussed in chapter 5 and 6. To assess the influence from taxation we have followed the same procedure for tax rates of 38% and 50%. The resulting gains differed in level from *MGAIN00*, but with pairwise correlation coefficients all above 0.997[102]. We conclude that *MGAIN00* will be a sufficient statistic for the after-tax refinancing incentive at least for the present sample of bonds[103]. Figure 7.2 displays *MGAIN00* for different bonds and settlement dates.

One should note that the exact level of the refinancing incentive is non-essential because a change in level will be reflected in new values for the estimated coefficients of the prepayment model. But the explanatory variables used for the estimation must of course correspond closely to the variables used in the pricing procedure.

---

[100] For further details see section 5.1.

[101] We have tried average gain as well, but that measure failed to detect the sudden increase in prepayments January 1989 and July 1991.

[102] This simply reflects that the sample consist of bond-loans only. For bonds issued under the cash-loan system one would have obtained a relatively larger reduction in gain for increased tax-rates. For these loans the use of an after-tax gain would probably increase explanatory power.

[103] A measure like $(B_{mt} - 100)/B_{mt}$ showed a correlation of 95% to *MGAIN00* indicating room for further simplifications of the model.

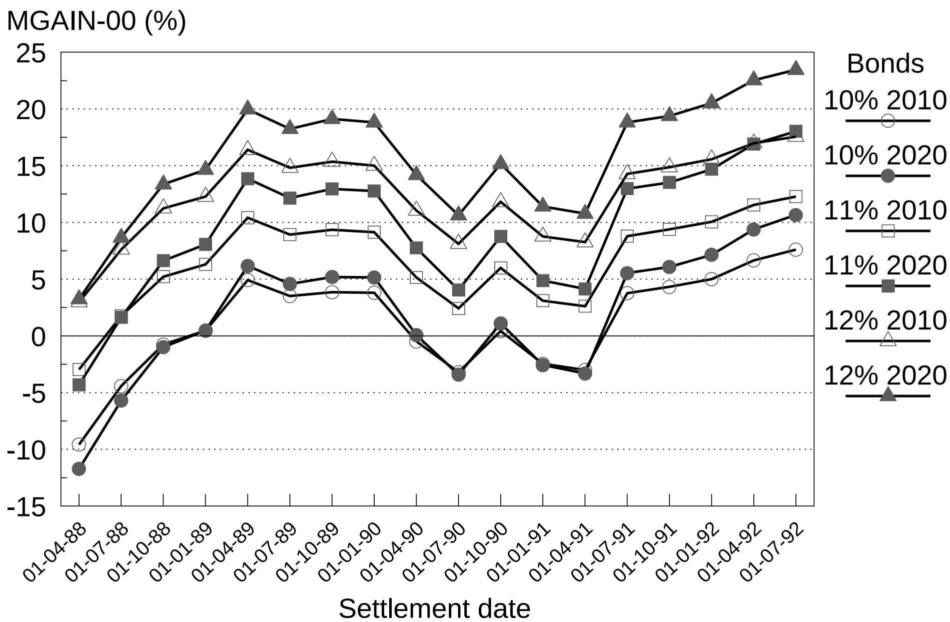


Figure 7.2: Examples of the pre-tax gain from pre-payment for 10, 11 and 12% MBBs maturing 2010 or 2020.

The second explanatory variable, *MATURITY*, equals the remaining time to maturity of the bond at each settlement date. From the analysis of optimal prepayment behaviour we would expect borrowers to require a higher gain from prepayment for longer term loans.

MBBs with a large share of corporate borrowers may react more quickly to changes in the level of interest rates. Some high-coupon MBBs issued after the 1989-period of high prepayment rates may furthermore contain a share of "speculative borrowers", typically corporations, who have taken a high-coupon mortgage with the intention of prepaying after a subsequent fall in interest rates. In the absense of published information from NYK and KD on the share of corporate borrowers a dummy variable, *HIGHRISK*, is created which captures newly issued 11% and 12% annuity bonds[104]. The high-risk dummy is shown in table 7.1.

To see if burnout effects are present in the Danish data we have followed Schwartz and Tourus(1989) and defined the variable *BURNOUT* equal to AO_{mt}/AO^*_{mt} where AO_{mt} is the amount outstanding from bond m at time t while AO^*_{mt} is the amount that would be present in the absence of prepayment. In the calculation we have assumed a starting

[104] Mortgages backing serial bonds are almost exclusively issued by households in connection with the so-called mix-loans. 30-year bonds, like 11% and 12% 2020 are not open for normal corporate issues, but they may contain loans to the agricultural sector. The bonds chosen have all been open for new issues August 1990 or later.

value of one in January 1988 for all bonds. The value of *BURNOUT* at the end of our sample period is shown in table 7.1. The record holder is the 12% 2010 annuity bond from NYK, with only 8.5% of the mortgages left. The two 11% 2012 bonds have experienced even higher prepayment rates, but they have not been open for the full sample period. For convenience we use the log transformation in the estimations, i.e. *LOGBURN=ln(BURNOUT)*.

Finally a constant equal to one, *CONST*, is included in the regressions to capture the level of mean required gain.

The explanatory variables can be subject to various transformations. However, the three simple specifications below seem to capture the data just as well as the more complex specifications tested[105].

Model A assumes a constant standard deviation of required gain, $\sigma_{mt} \equiv \sigma$, with mean required gain being a linear function of time to maturity. The dummy variable *HIGH-RISK* affects prepayment through a different level of the mean, i.e.

$$
\begin{aligned}
g^*_{mt} &= a_1 \cdot CONST + a_2 \cdot MATURITY_{mt} + a_3 \cdot HIGHRISK_{mt} \\
\sigma_{mt} &= \sigma
\end{aligned}
\tag{7.8}
$$

The following relation shows the correspondence between σ, a_1, a_2, a_3 and the β-coefficients of the general probit model[106]

[105] Models in which the standard deviation depends linearly or by the square root of time to maturity, i.e. $\sigma_{mt} = \sigma \cdot T_{mt}$ and $\sigma_{mt} = \sigma\sqrt{T_{mt}}$ have been tested with almost equivalent results. In these models the constant part of mean required gain often turns out insignificant. Models in which *HIGH-RISK* and *LOGBURN* affects σ_{mt} has also been tried with less success.

[106] Subscript *mt* has been suppressed.

$$
\begin{aligned}
&\text{Prob}[y_i = 1] \\
&\quad = \Phi((MGAIN00 - g^*)/\sigma) \\
(7.9) \qquad &\quad = \Phi\left(\frac{1}{\sigma}MGAIN00 - \frac{a_1}{\sigma}CONST - \frac{a_2}{\sigma}MATURITY - \frac{a_3}{\sigma}HIGHRISK\right) \\
&\quad \equiv \Phi(\beta_1 \cdot MGAIN00 + \beta_2 \cdot CONST + \beta_3 \cdot MATURITY + \beta_4 \cdot HIGHRISK)
\end{aligned}
$$

Model B extends model A with the *LOGBURN* variable which is added in the same way as *HIGHRISK*. Model C is equal to B except for a special dummy variable *VERYRISK*, which equals one for the two 11% 2012 bonds and zero otherwise. Correspondingly *HIGHRISK* is set to zero for these two bonds.

Table 7.2: *Parameter estimates for three different prepayment models.*

	Model A	*Std.Err.* *T-ratio*	Model B	*Std.Err.* *T-ratio*	Model C	*Std.Err.* *T-ratio*
MGAIN-00	0.0987	*0.0223* *4.4240*	0.1303	*0.0330* *3.9480*	0.1286	*0.0329* *3.9060*
CONST	-1.9006	*0.4870* *-3.9030*	-1.5275	*0.5440* *-2.8080*	-1.6778	*0.5661* *-2.9640*
MATURITY	-0.0531	*0.0241* *-2.2030*	-0.0799	*0.0316* *-2.5290*	-0.0724	*0.0321* *-2.2580*
HIGH-RISK	0.5181	*0.2429* *2.1330*	0.6755	*0.2667* *2.5320*	0.5489	*0.2826* *1.9420*
LOGBURN			0.4535	*0.3074* *1.4750*	0.4137	*0.3091* *1.3380*
VERY-RISK					1.2027	*0.4399* *2.7340*
R^2	0.4638		0.5370		0.6092	
AIC	74.2885		74.1434		74.1388	

Maximum likelihood estimates of β-coefficients for the three models with asymptotic standard errors and T-ratios are shown in table 7.2 while table 7.3 contains the implied values of σ, a_1, a_2, a_3 etc. The explanatory variables of Model A are all significant. Mean required gain (MRG) increases as expected with time to maturity. A normal 20 year MBB have a mean of 30.01% and a standard deviation of 10.13%. A pre-tax gain of 15% would thus induce an estimated 6.91% prepayment rate from a normal 20 year bond. High-risk bonds have a 5.25% lower MRG and therefore higher prepayment rates.

Table 7.3: *Estimated parameters of the required gain distribution.*

	Model A	Model B	Model C
Std.dev., σ	10.1313	7.6751	7.7757
Mean-Gain (a_1)	19.2555	11.7234	13.0463
Maturity (a_2)	0.5377	0.6134	0.5629
Mean, 20 year bond	30.0093	23.9920	24.3047
High-Risk (a_3)	-5.2488	-5.1843	-4.2680
LogBurn (a_4)		-3.4805	-3.2170
Very-Risk (a_5)			-9.3522
	Prepayment rates at 15% gain		
20 year normal bond	6.91%	12.08	11.57
20 year, high risk bond	16.75	31.01	25.85
20 year, high risk, BURNOUT=0.5	16.75	20.90	17.50

The burnout effect of model B is not significant, but the estimation nevertheless shows some interesting changes. Estimated prepayment rates for bonds with no burnout (*BURNOUT=1*) are high compared to model A. This reflects that some of the relatively low prepayment rates could be explained by the burnout variable, while Model A takes such rates as a sign of lower prepayment rates in general.

Model C includes the *VERYRISK* adjustment for the two 11% 2010 bonds. This turns out significant, but now the *HIGHRISK* variable becomes insignificant. The *LOG-BURN* variable is not significant in Model C either[107].

[107] The explanatory variables *BURNOUT*, *MGAIN00*, *HIGHRISK* and *VERYRISK* are obviously not uncorrelated, which explains why the estimated parameters differs across models.

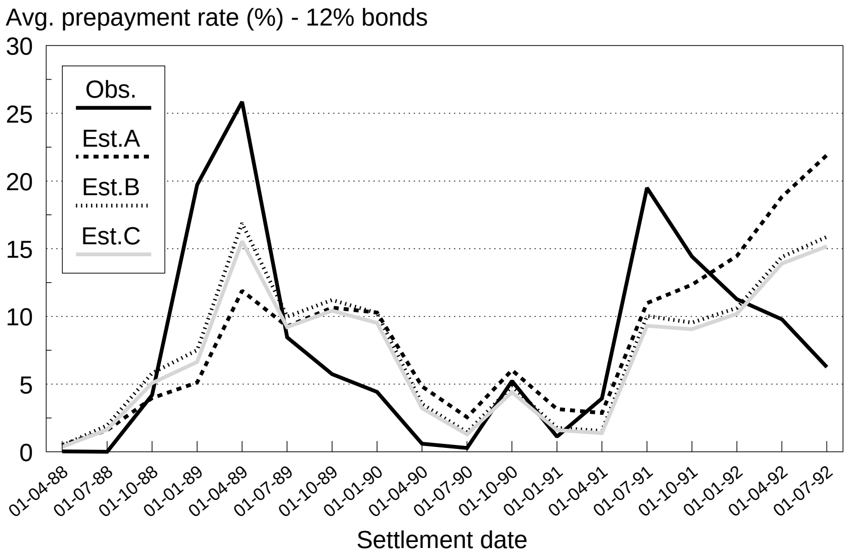
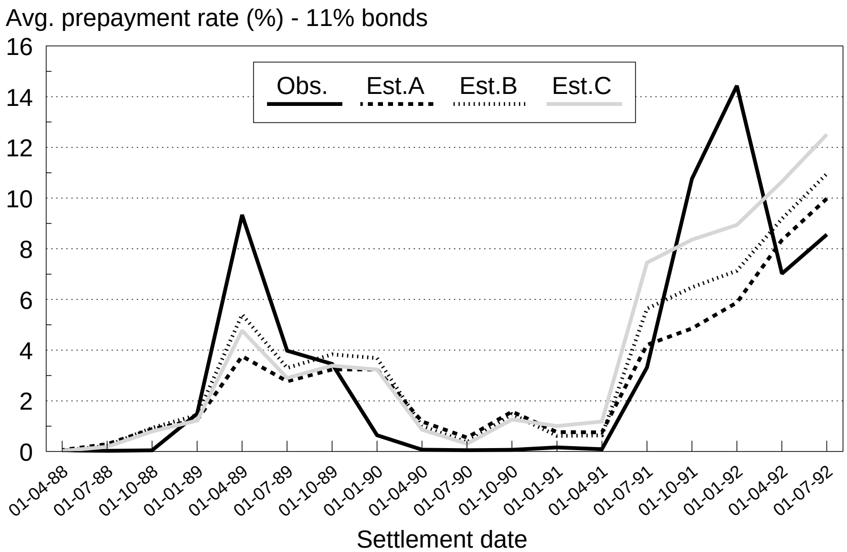


Figure 7.3 (a)-(d): Comparison of observed and estimated average prepayment rates as a function of the settlement date for different coupon rates.

(a) Coupon 12%



(b) Coupon 11%

Predicted prepayment rates from the three models for different settlement dates and averaged across coupon rates are shown in figure 7.3 (a-d). The models are very close for 9% and 10% bonds, but differences arise for the 11% and 12% issues. Data seem to contain more intertemporal variation than predicted by the models. Some borrowers react immediately when interest rates fall, but the high prepayment rates fade out quickly. The burnout variable of model B and C helps to explain this phenomenon, but a burnout-measure like the one proposed by McConnell and Singh(1991), cf. section 7.1.4, might be more appropriate. In their model prepayments rise if the current rate falls below the minimum rate for the last 12 months. An even more promissing sugges-tion could be to use the change in gain as an explanatory variable.

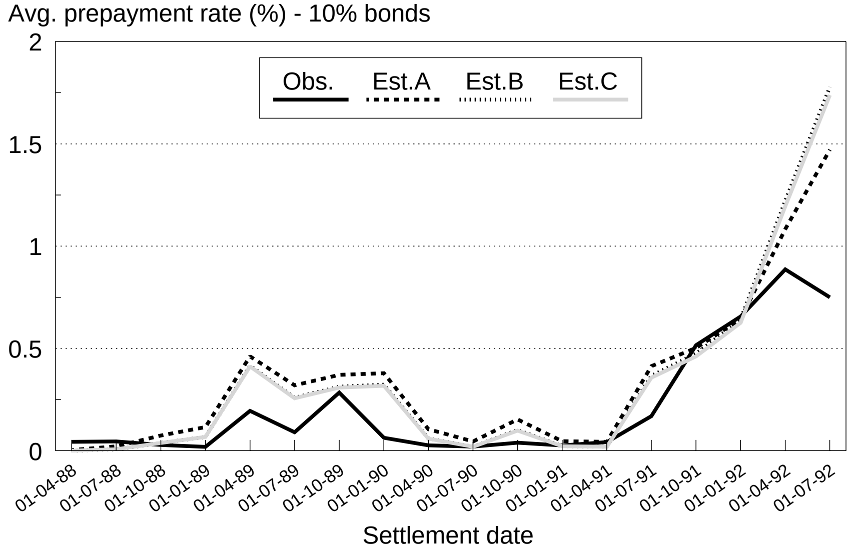
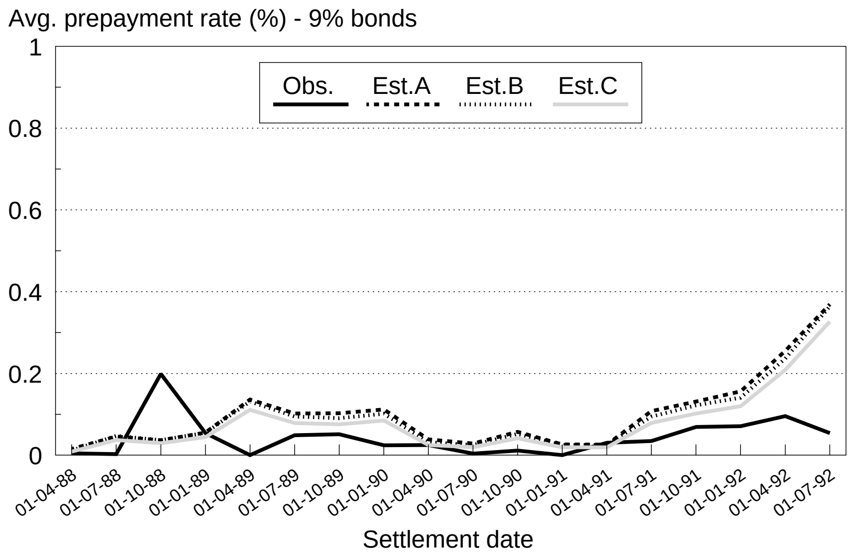


Figure 7.3-(c) Coupon 10%



(d) Coupon 9%

All three models overstate prepayment rates for the last part of the sample period. Several explanations might be given. The first could be that borrowers react to a sudden increase in gain but not to a steadily increasing one. The second is the use of a non-callable term structure to calculate gain. If borrowers use 9% MBBs as their refinancing alternative they will tend to underestimate the fall in the market rates of interest due to the counteracting influence from the prepayment option of the 9% bonds. The third explanation could be that many borrowers (wrongly) expected a sharp fall in interest rates following the outcome of the EEC referendum and therefore postponed the prepayment decision.

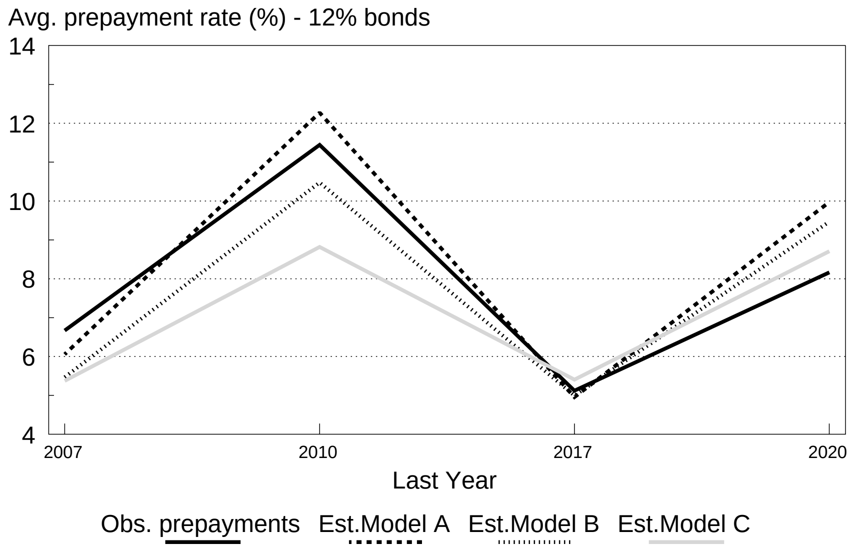
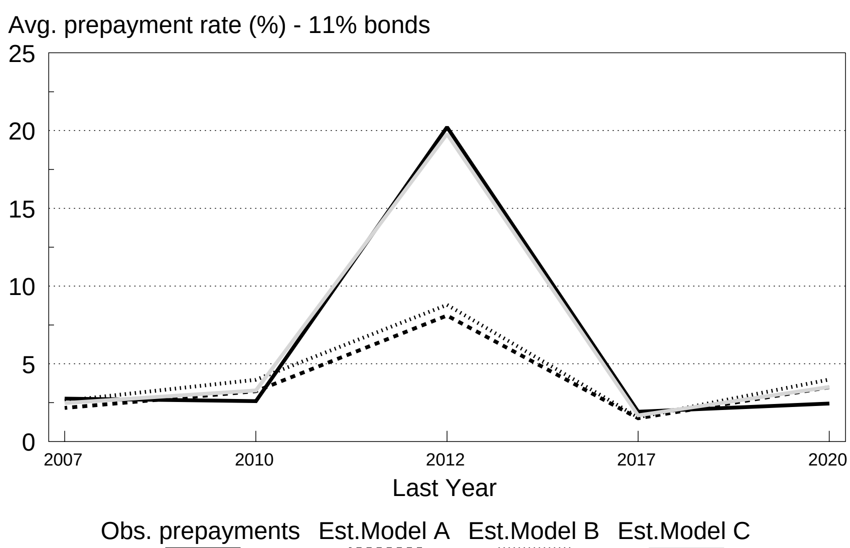


Figure 7.4: (a)-(d) Comparison of observed and estimated average prepayment rates as a function of last payment year for different coupon rates.

(a) 12% MBBs

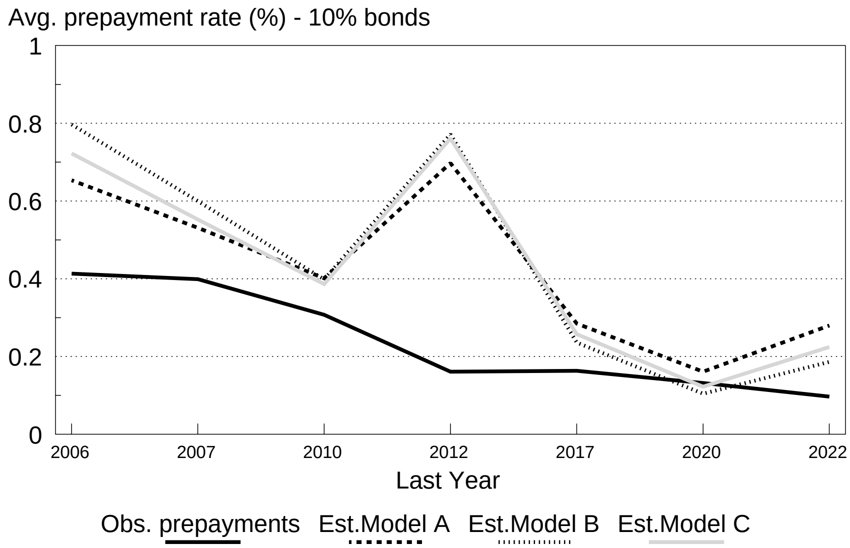


(b) 11% MBBs

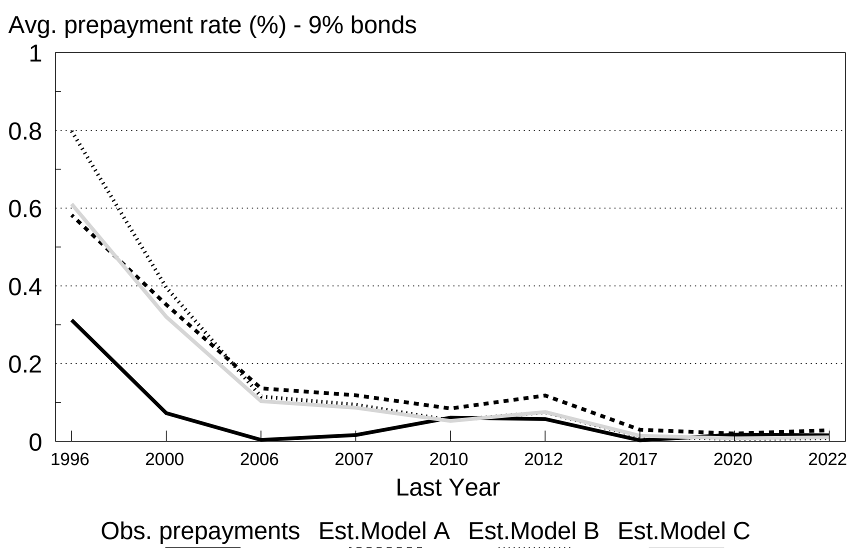
Figure 7.4 compares observed and predicted average prepayment rates for different maturities. The maturity dependency seems to be very well explained by the models, except for the two 11% 2010 bonds which require the special treatment of model C.

The choice between the different models is not a simple matter. Table 7.2 shows R^2 from the regression of λ_{mt} on $\Phi(\beta' x)$. Model C performs best by explaining more than 60% of sample variance. The probit model is a heteroscedastic model and the use of R^2

Figure 7.4:
(c) 10% MBBs



(d) 9% MBBs



is questionable, cf. Amemiya(1981)[108]. Amemiya prefers the so-called Akaike Information Criterion, AIC, which adjusts for heteroscedasticity as well as the number of parameters used. AIC is defined as minus the log likelihood plus the number of parameters. In general one should choose the model with the lowest AIC. This points to model B or C but now with model A close behind. Model A is the only model in which all included effects come out significant and the increased level of explanatory power of model B and C might be an in-sample phenomenon. At the present state of analysis we thus prefer the less sophisticated and hopefully more robust model A.

---

[108] Observations close to the mean have the highest variance and sole reliance on R^2 might bias the model towards an explanation of these observations.

The current section has presented an empirical model for Danish prepayment data. The model was based on a newly constructed data set and the model still needs refinement. The results are nevertheless quite promising. Model A summarized more than four years of prepayment experience in a simple four parameter model and we have shown how small modifications might increase explanatory power considerably.

No significant path-dependent burnout effect was found, but that will most likely change when new specifications and more data become available. We have identified some bonds with high or very high prepayment risk. The inclusion of dummy variables to explain particular bonds is not satisfying and one could hope that the mortgage institutions change their current practice and make data on the composition of borrowers in the individual issues available to the investors. This would probably remove a large part of the remaining uncertainty regarding mortgage backed bonds.

7.4 Pricing results

The estimated prepayment function can be used in many different settings, but our main ambition will be to predict and price the future cash flow from a MBB. We have developed a full PC pricing model for Danish MBBs based on the BDT-model discussed in chapter 6. The current section shows empirical results from this model for the period January 1988 to July 1991[109].

The pricing of MBBs on a single trading date is a multi-stage procedure starting with an estimation of the zero-coupon term structure. The following analysis uses the NC-FS model discussed in chapter 2. This is a flat-spline model based on a sample of large non-callable bonds.

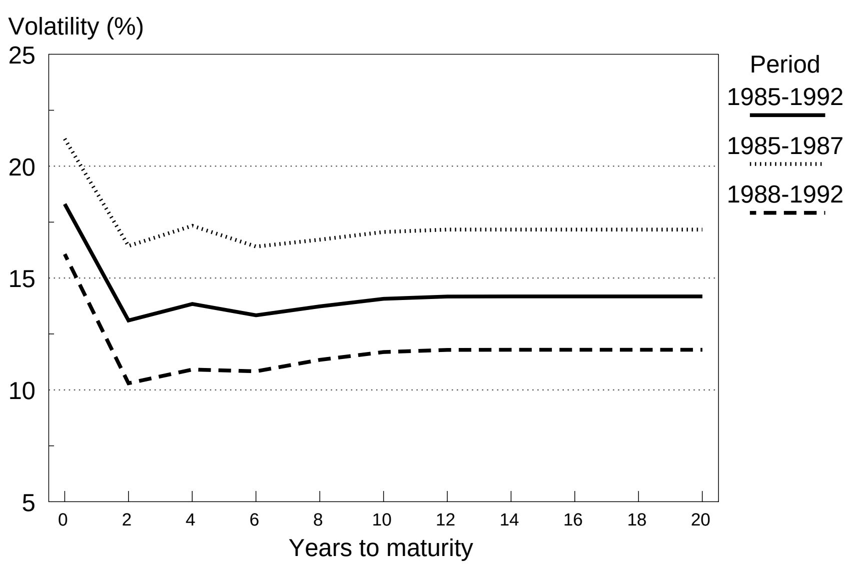


Figure 7.5: Estimated volatility for zero coupon yields as a function of time to maturity. Yield-estimates are taken from the non-callable flat-spline model.

The next step will be to find an estimate of future volatility. Any time dependent function of future short term volatility can be used in the model. The structure of future short term volatilities determines the models degree of mean reversion of future interest rates, cf. Jamshidian(1991) or Hull and White(1990a). Figure 7.5 shows estimated volatility for different zero coupon yields as a function of time to maturity. The results are based on weekly term structure estimates for the full period January 1985 to July 1992 as well as two subperiods, 1985-1987 and 1988-1992. The plot

[109] Computations of prices and convexities were done with the program RIO/Optikon developed by the author, cf. the footnote in chapter 6. Data on bonds and prices have been drawn from the Financial Database of the Aarhus School of Business.

indicates a trend towards a lower volatility during the sample period. The last period have witnessed short term yield volatilities around 15% while longer term volatilities converge toward 12%. These findings are very similar to the ones obtained by Visholm(1992) on the March 1989 to April 1991 period. The choice of future volatility may have a considerable impact on the pricing results[110], but we have not investigated this issue further. The following analysis assumes a constant 14% volatility of future short term interest rates. This estimate is used throughout the sample period.

The estimated term structure and the volatility are used as input to the calibration procedure of section 3.1.4 which results in a derived lattice for the future short term interest rates. A shifted lattice, used for the calculation of level-duration, is calculated next as discussed in section 6.4. The results are stored for repeated use in the following.

The pricing procedure is done for a list of MBBs using the approach of section 6.2. For each MBB scheduled payments are calculated and the backwards pricing procedure starts at maturity. At each date-event (n,s) the program calculates the hold-on value of the bond, $V^{+}(n,s)$, the value of the bond in case of prepayment, $W(n,s)$, the value of scheduled payments from the underlying non-callable mortgage, $B_m(n,s)$, and the prepayment value of the mortgage, $W_m(n,s)$, with respect to transaction costs of 2% and a 65 days term of notice. Percentage pre-tax gain from prepayment is found as $g_m(n,s) \;=\; 100 \cdot (B_m(n,s) - W_m(n,s))/B_m(n,s)$ which corresponds closely to the variable *MGAIN00* used in the estimation of the prepayment function[111]. The variable $T_m(n,s)$ is calculated as the remaining number of years to maturity at (n,s). Using model A of the previous section we find the prepayment rate at date-event (n,s), $\lambda(n,s)$ as[112]

[110] The future short term interest rate volatilities can be derived from the initial term structure of volatilities by an extension of the calibration procedure of section 3.1.4. The estimation of a volatility structure for the BDT-model raises some statistical problems, because long term yields are not log-normal distributed and the structure of volatilities cannot be constant, cf. Jakobsen and Jørgensen (1991).

[111] If the prepayment function had been estimated on after-tax gain we would use the after-tax procedure of section 4.3. This adds a small initial step which calibrates the after-tax lattice to the current after-tax term structure. Computation time for each bond increase by approximately 50% in the current implementation when after-tax calculations are used.

[112] The approximation of $\Phi(\cdot)$ was taken from Abramowitz and Stegun(1970, p. 932, 26.2.16). A logit model may be used as a substitute for the probit, which could diminish computation times due to the simpler distribution function.

(7.10) $$\lambda(n,s) = \Phi(0.0987 \cdot g_m(n,s) - 1.9006 - 0.0531 \cdot T_m(n,s) + 0.5181 \cdot H_m)$$

where $H_m = 1$, if the MBB belongs to the high-risk group and $H_m = 0$ otherwise. The bond value is finally found as $V(n,s) = \lambda(n,s)W(n,s) + (1-\lambda(n,s))V^{+}(n,s)$ and the procedure moves to the next date-event. The proces is repeated for each date-event down to $(0,0)$ which results in a theoretical value for the callable mortgage backed bond at time zero. The computation of level-duration requires a similar calculation for the shifted lattice[113].

We have applied the valuation procedure on 59 Wednesdays with intervals of four weeks for the period January 1988 to July 1991. On each day the theoretical price and level-duration was calculated for the all 33 NYK MBBs shown in table 7.1 above. A range of other statistics was calculated as well, some of which are used in the next section.

Figure 7.6 plots estimated price against market price for all 59 dates in the sample. Prices have been averaged across all MBBs having the same coupon rate. The estimates from the model follow actual market prices quite closely. One should consider that these estimates are derived with no input whatsoever from actual MBB-price behaviour. Only volatility and term structures estimated on non-callable bonds as well as the four parameters of the prepayment function are used to guide the estimates.

[113] The calculations shown below was done with 4 steps per year which corresponds to 7.260 date-events for a 30 year mortgage. Each valuation including the duration step took 5 seconds on a 25 MHz 386 PC equipped with a mathematical coprocessor. Computation times include some overhead as the program calculates several derived statistics and saves all results in a database. Experiments with 8 steps per year gave almost exactly the same pricing results.

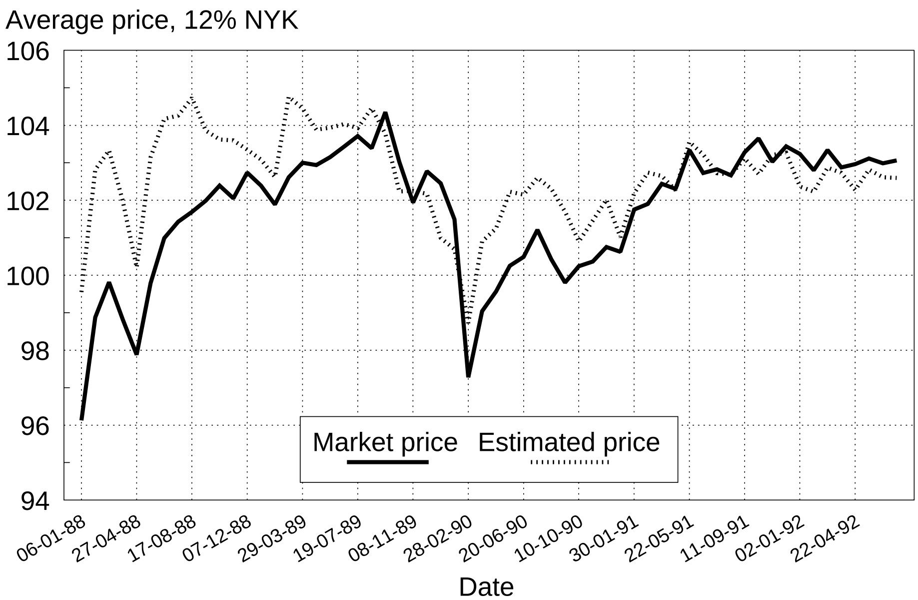
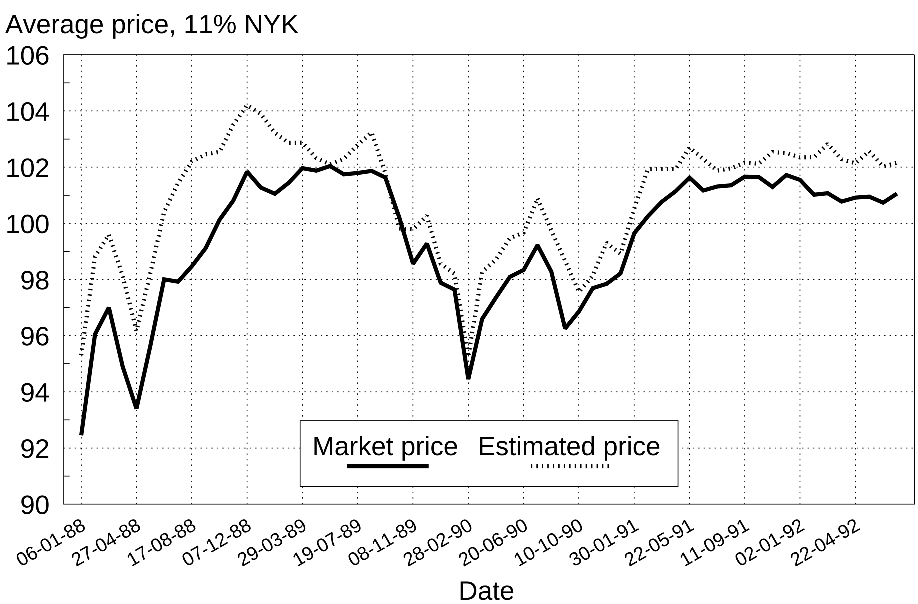


Figure 7.6 (a)-(d): Plot of average market price against average estimated price for all 59 Wednesdays with averages taken across MBBs having the same coupon rate.

(a) Coupon 12%

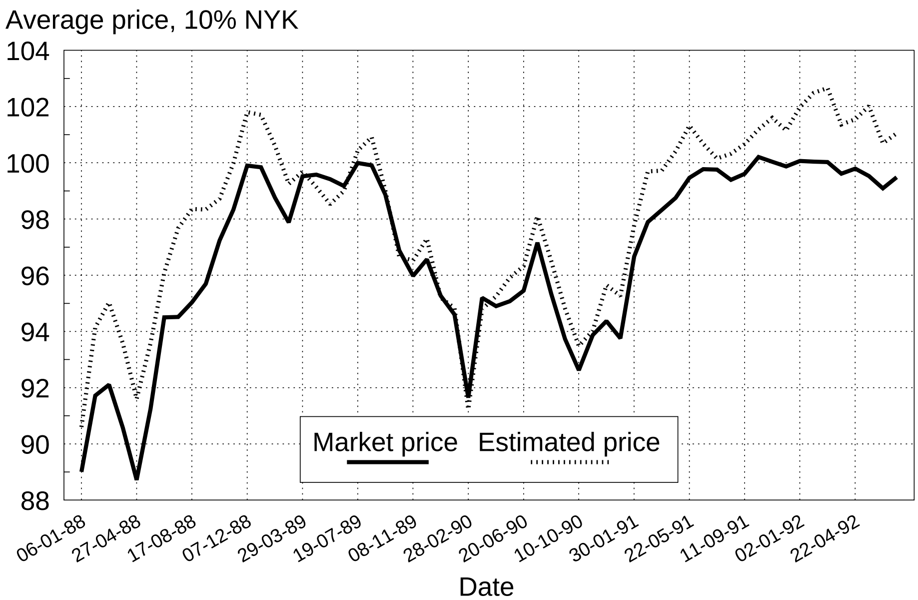


(b) Coupon 11%

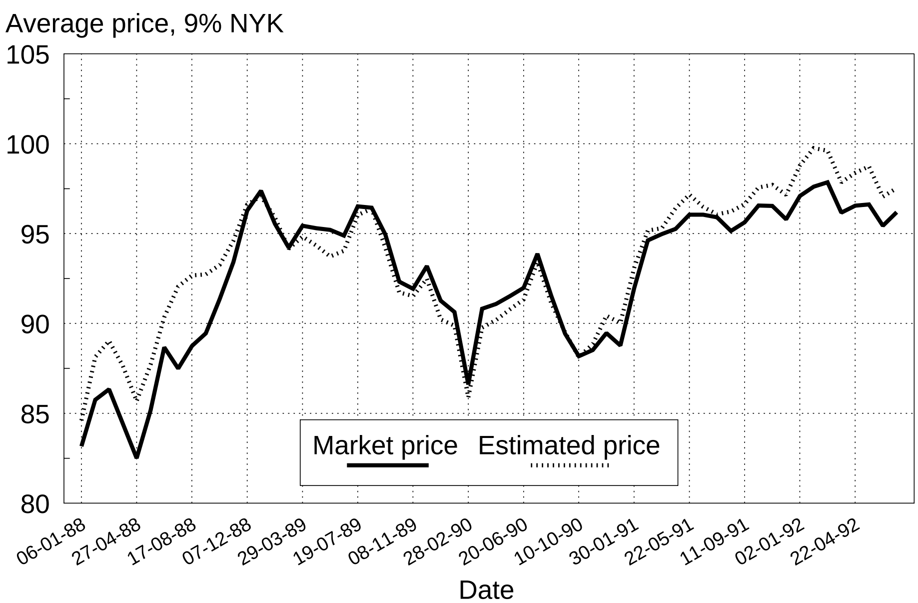
Estimated level-duration, i.e. the percentage fall in estimated value from an upward parallel shift in the yield curve, is shown for all 2010 MBBs in figure 7.7. The plot conveys the highly coupon rate dependent nature of this measure. Duration is lowest for the 12% MBBs with estimates of almost zero duration in the last part of the sample period. Judging from the price plots this seems to capture the actual market price volatility quite well.

Figure 7.6

(c) Coupon 10%



(d) Coupon 9%



To be critical our model seems to over-value MBBs relative to the market, especially in the first half of the period. This is confirmed from the bar-chart of figure 7.8 which shows average net present value, defined as the difference between estimated price and market price. Average over-valuation range from zero to almost two price points.

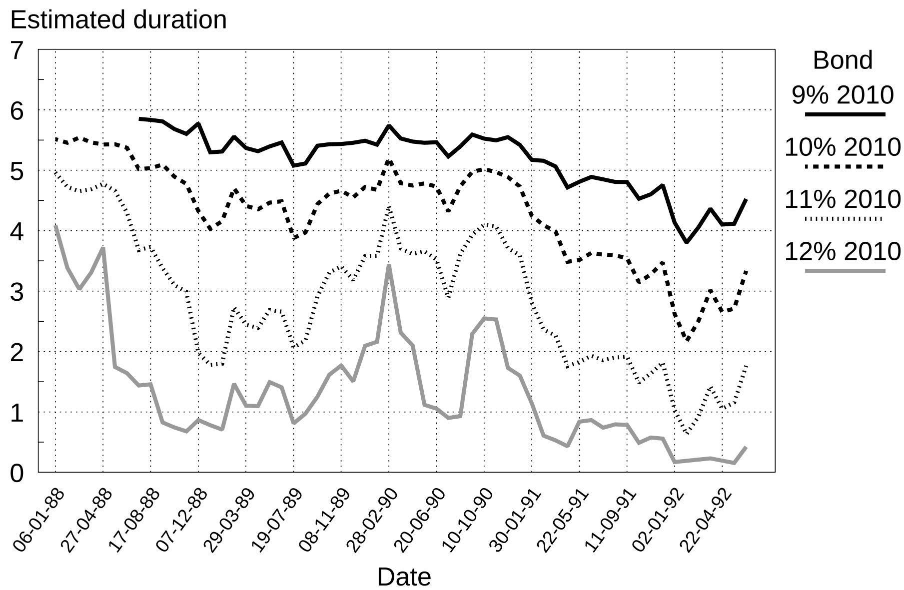


Figure 7.7: Estimated average level duration for different dates. Averages are taken accros 2010 bonds having the same coupon rate.

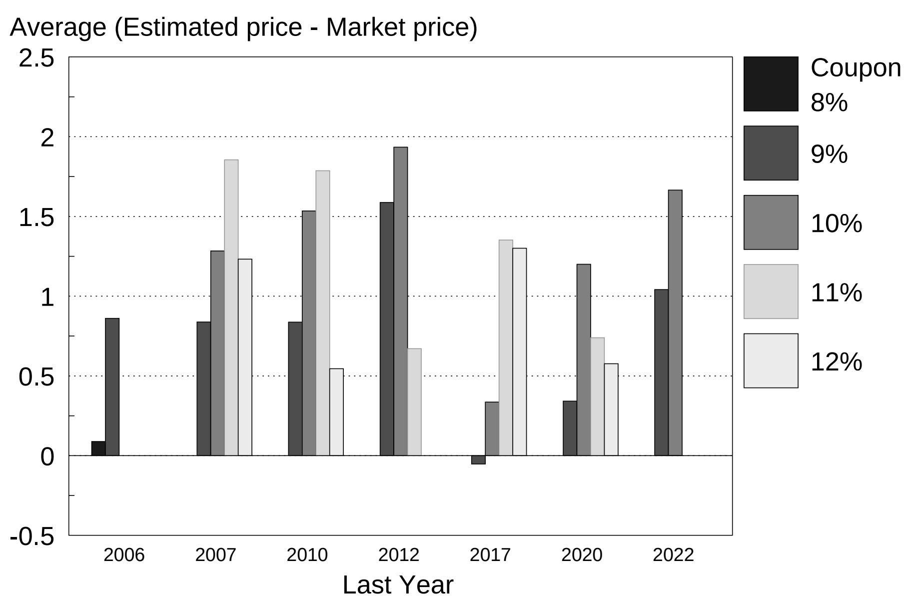


Figure 7.8: Net present value for 33 NYK issues averaged across all dates and all bonds having the same coupon rate and closing year.

Several factors may explain the over-valuation. The most obvious is a change in implied volatility. As seen from figure 7.5 the market have witnessed a fall in volatility and it would be fair to assume that implied volatility was higher for the first part of the period, which would reduce the price estimates, cf. section 6.3.

The prepayment function could also explain some of the differences. Model A underestimates the sudden increase in prepayments early 1989 as shown above and estimated prices will therefore tend to increase more than adequate after the fall in interest rates. This also explains why overvaluation disappears in the period from mid 1989 to mid 1990 in which model A slightly overestimates actual prepayment rates.

The final explanation could be that the market prices are wrong. This is the subject of the next section.

7.5 An Analysis of Holding Period Returns, 1988-1992.

We have now developed a full pricing model of Danish callable mortgage backed bonds. The model is based on a ’micro-economic’ decision model for the individual mortgage holder. It was shown how the parameters of this behaviorial model can be inferred from observed prepayment data. The estimated prepayment function has been combined with a general option valuation model, which takes the zero coupon yield curve estimated on a sample of non-callable bonds together with an estimate of future volatility as input. From this combined model prices and duration measures were calculated for a sample of MBBs in the period 1988 to 1992. The pricing results corresponded rather closely with actual market prices although some evidence of a pricing bias was given.

This final section investigates the return characteristics of the model. The ability to predict and/or explain the structure of future returns is one of the ultimate tests of a financial pricing model. Close adherence to observed market prices is only a secondary objective as market prices may be wrong.

To test the performance of the model one could simulate several trading strategies based on the differences between estimated and actual market prices. The problem with this approach is that each simulation works like an index, aggregating one particular aspect of the data. The following analysis employs a regression model similar to the multi-factor models of stock market returns. In the period from January 1988 to July 1992 we have calculated a data-set consisting of four-weekly holding period returns (HPR) for a large number of bonds. Each HPR is combined with a number of explanatory variables. Finally a regression model is used to analyse how mispricing, duration etc. affect return. The basic idea is that any effect not present in these data is probably uninteresting for the investor. The examples given are very preliminary but they will hopefully indicate the power of this approach.

To compute returns we have developed yet another PC-program. The program takes a list of bondcodes as well as a list of return periods as input and outputs a database containing holding period returns together with other statistics for all bonds in all periods.

The program draws on several databases. The database, *PRICES*, contains daily price quotations for the bonds. Another set of databases, *BONDS*, summarizes the necessary information to calculate scheduled payments for each bond. Prices and bond information were drawn from the financial database of the Aarhus School of Business. Money market interest rates for the period have been kindly provided by Bikuben A/S. Finally the program needs a database, *DRAWINGS*, containing the actual repayment rates collected for the analysis of section 7.2.

The calculation of HPR can be described as follows. Consider a single period from trade date T_1 to T_2. Start and ending period market prices P_1 and P_2 are obtained from *PRICES*. To increase the number of return periods we have included prices quoted up to seven days prior to each trade date. For each period the reinvestment rate is determined as the money-market rate of interest at T_1. The dates at which transaction at T_1 and T_2 becomes effective, S_1 and S_2, as well as accrued interest, A_1 and A_2, are calculated according to the Danish bond market conventions. Next the program calculates the payments from the bond in the period. For non-callable bonds this simply amounts to scheduled coupon payments and repayments on principal, while MBB-calculations uses the actual repayment rates taken from *DRAWINGS*. Repayments on principal are allocated to the period in which the drawings become published, which may be 1-3 months ahead of the final settlement date.

The calculations assume each bond to be bought at T_1 and sold at T_2. Total end-of-period value, V_2, is found as the sum of $P_2 + A_2$ multiplied by remaining principal plus the value of any coupons and repayments due in the period. We add interest earned from reinvestment of payments. This adjustment may become negative if the repayments are due after S_2. HPR for the period is found as $(V_2 - P_1 - A_1)/(P_1 + A_1)$. Period HPR is converted into an annualized return by the multiplication of 360 divided by the number of interest days between S_1 and S_2[114].

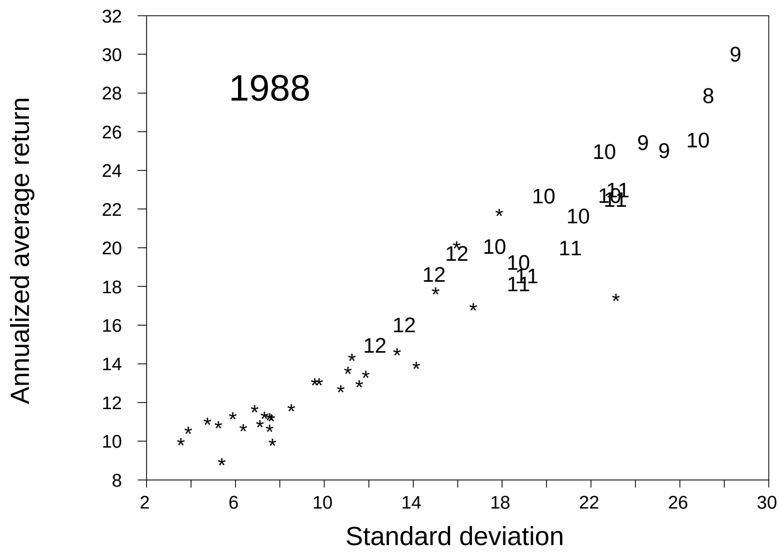
[114] HPR could be computed for indexed linked bonds and adjustable rate bonds as well.

The basic data-set used in the following analysis contains holding period returns (HPR) for all four weeks periods between January 1988 and July 1992. The list of bondcodes includes the 33 NYK MBBs analysed in the previous section as well as all large non-callable government bonds. The appendix contains a summary of the basic data-set. For each bond the listing shows average annualized return for each year in the sample period, provided returns could be calculated for all 4 weeks periods. The listing furthermore contains the average reinvestment rate, the standard deviation on monthly annualized returns and the risk adjusted excess return. Finally starting year values of yield-to-maturity and Macaulay duration calculated on scheduled payments are given.

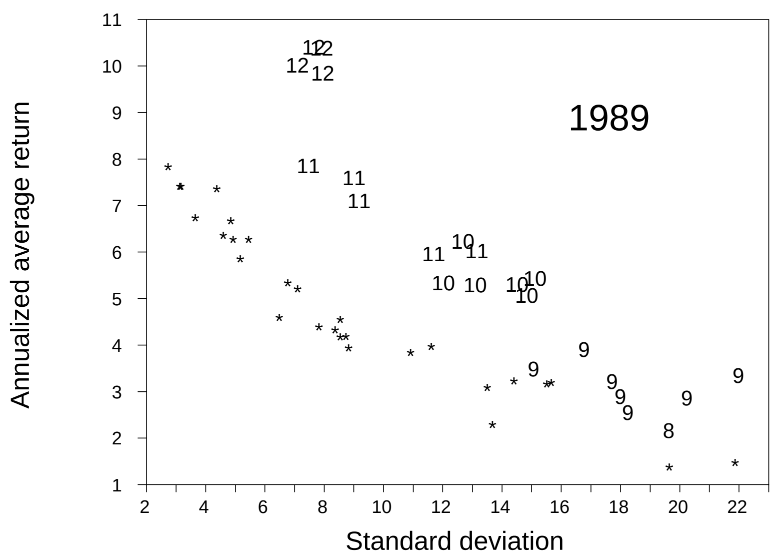
A full analysis of the data-set is outside the scope of the present thesis. Instead we show a few results in order to evaluate the pricing model of the previous section. It was shown in the previous section our model ’overvalued’ MBBs relative to market prices especially in the 1988 to 1989 period. For 11% and 10% bonds overvaluation returned from 1991. The pricing model dynamically compares MBBs to non-callable bonds of similar risk. If the model is correct, we would expect bonds with positive net present values[115] to be relatively cheap, i.e. market prices are too low, and the MBB should therefore obtain a high return relative to similar non-callable bonds.

[115] Net present value (NPV) is defined as present value less market price. Positive NPV indicates a ’cheap’ bond and vice versa.

Figure 7.9: Plot of average HPR against the standard deviations of HPR for 1988-1992. Non-callable bonds are marked by a '', while MBBs are marked by their coupon rate.*

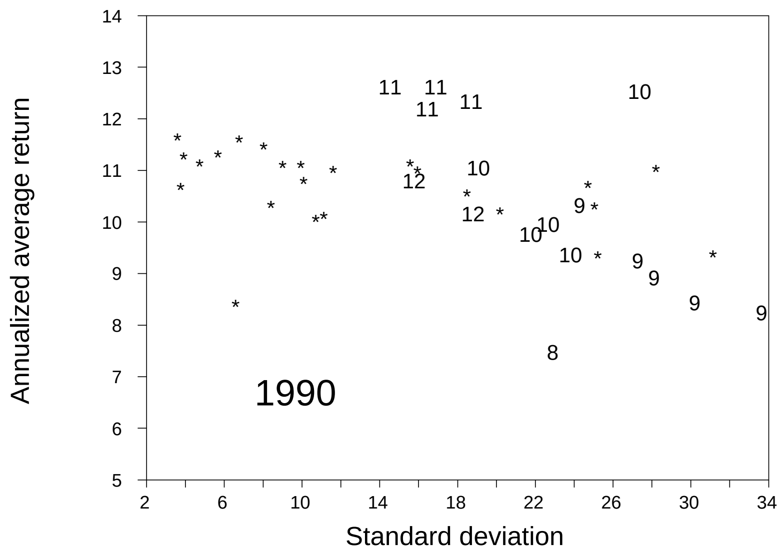


(a) 1988

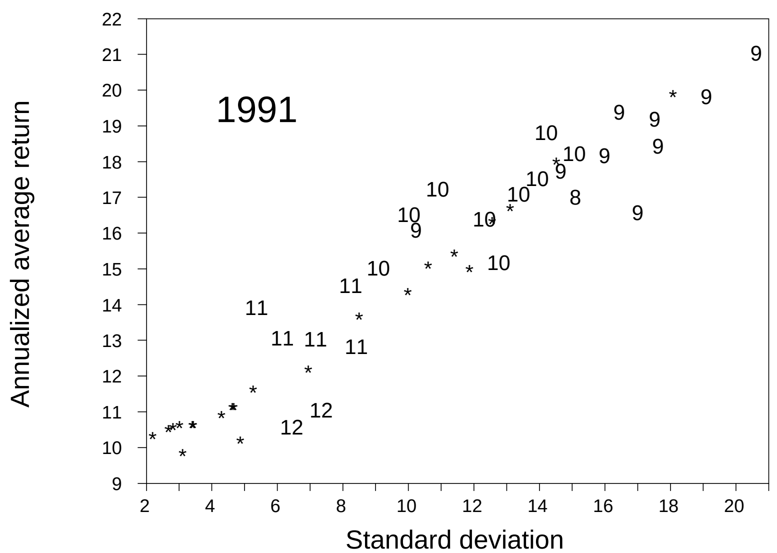


(b) 1989

(c) 1990

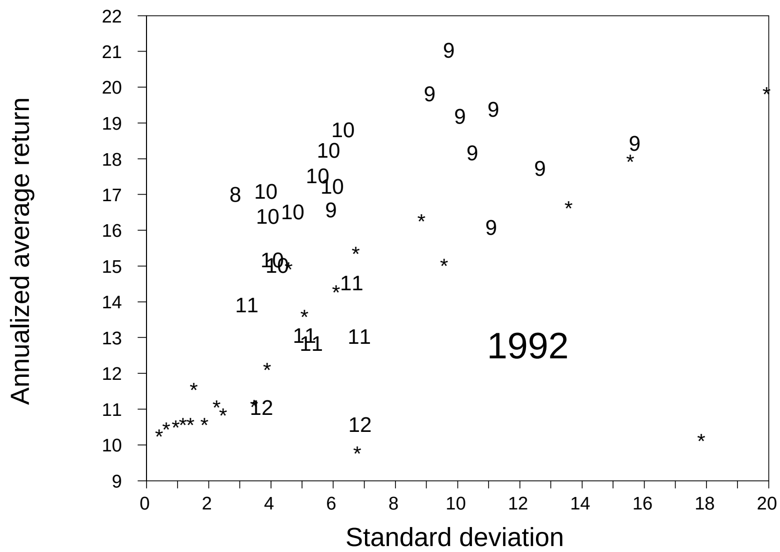


(d) 1991



To investigate these issues figure 7.9 (a)-(e) show average HPR against its standard deviation for all bonds in the sample. Standard deviation is taken as a fundamental indicator of return risk. Computations are done for each of the five years 1988-1992. The plots should be compared to the price plots 7.6 above.

(e) 1992



On average the valuation of the model performs remarkably well. No significant deviations are seen in 1988. This would be expected as any excess return from buying a relatively cheap bond will be small compared to the return following a sharp fall in interest rates. Returns from 12% and 11% MBB are far above comparable non-callable bonds in 1989. This corresponds to a period where MBBs change from positive to zero NPV, which means that the bonds obtain a positive price adjustment on top of high coupon payments. 11% bonds continue with high HPR in 1990, while the 12% bonds fares like the non-callable bonds. The 1991-1992 plot shows below average performance from 12% bonds which now have zero or negative NPV. 11% bonds does well and the 10% MBBs with high positive NPV have correspondingly high HPR.

The interpretation of plots is highly subjective and the aggregation of data into yearly averages conceals lots of information. The last part of our analysis therefore turn to the regression approach mentioned above.

The preliminary regression model used below is inspired by the well-known approximative return formula by Babcock(1984)

(7.11) $$HPR = Y_1 + \left(1 - \frac{D_1}{\Delta t}\right) \cdot (Y_2 - Y_1)$$

Babcocks formula states that annualized holding period return for a period of lenght Δt equals starting yield to maturity, Y_1, plus the 'unexpected' price increase or capital gain in the period. The price increase is found as the change in yield to maturity multiplied by a term which include minus the initial Macaulay duration, D_1 adjusted to provide annualized changes.

The basic data-set consists of 3577 individual return observations. For each observation we have collected a number of explanatory variables. *NCYTM* equals percentage yield to maturity of the scheduled payments provided the bond is bought at the equilibrium prices found from the non-callable term structure. *NCYTM* is given on a monthly compounded basis. *NCYTM* can be seen as a very rough measure of promised yield[116]. *DELTAYTM* equals the change in *NCYTM* during the period i.e. $NCYTM_2 - NCYTM_1$. We use *DELTAYTM* as a proxy for any change in the level of interest rates. *NCYTM* and *DELTAYTM* are both found from the term structure of non-callable bonds using no knowledge of actual market prices. For all non-callable bonds in the sample *NCVOLA* is defined as *DELTAYTM* multiplied by the Fisher-Weil level-duration measure (6.4). We expect that *NCVOLA* will be roughly proportional to the capital gain part of HPR. *NCVOLA* equals zero for MBBs. In a similar manner *MBBVOLA* is defined as *DELTAYTM* multiplied by the level-duration measure (6.12) calculated from the MBB-pricing model. *MBBVOLA* equals zero for non-callable bonds. *MBBVOLA* should explain returns similar to *NCVOLA* provided the MBB-model is correctly specified.

To investigate possible effects from an initial mispricing of the bond we define *NCNPV* as the relative percentage difference between present value and market price for non-callable bonds. *MBBNPV* is defined in a similar way as the relative percentage difference between the estimated price from the MBB-model and the market price. *NCNPV* equals zero for MBBs while *MBBNPV* equals zero for non-callable bonds. It would be expected positive NPV leads to higher HPR. We finally employ a constant variable *ONE* equal to one and three dummy variables *MBB10*, *MBB11* and *MBB12* each of which equals one, if the bond is a 10%, 11% or 12% MBB respectively, and equals zero otherwise.

Estimation results from three different models are shown in table 7.4. Model X is given by

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[116] Theoretical inclined readers may prefer to skip directly to the results.

$$HPR = \beta_1 \cdot ONE + \beta_2 \cdot NCYTM + \beta_3 \cdot NCVOLA + \beta_4 \cdot NCNPV + \beta_5 \cdot MBBVOLA + \beta_6 \cdot MBBNPV + \varepsilon \tag{7.12}$$

Model Y excludes the constant *ONE* while model Z includes *MBB10*, *MBB11* and *MBB12*. All models have been estimated by ordinary least squares.

Table 7.4: *Parameter estimates for three different OLS regression models.*

	Model X	*Std.Err.* *T-ratio*	Model Y	*Std.Err.* *T-ratio*	Model Z	*Std.Err.* *T-ratio*
ONE	-1.10040	*1.75244* *-0.628*				
NCYTM	1.06347	*0.18571* *5.2727*	0.947247	*0.01519* *62.363*	0.971238	*0.01660* *58.508*
NCVOLA	-10.9844	*0.19206* *-57.193*	-10.9999	*0.19044* *-57.761*	-10.9857	*0.19018* *-57.764*
NCNPV	2.45919	*0.33080* *7.434*	2.45151	*0.33055* *7.416*	2.46915	*0.33007* *7.481*
MBBVOLA	-8.42454	*0.12388* *-68.006*	-8.44025	*0.12132* *-69.573*	-8.44244	*0.12113* *-69.695*
MBBNPV	2.25615	*0.12172* *18.536*	2.25409	*0.12166* *18.527*	2.50164	*0.13957* *17.924*
MBB12					-1.05121	*0.54825* *-1.917*
MBB11					-1.6252	*0.50053* *-3.247*
MBB10					-0.96083	*0.41708* *-2.304*
R^2	0.7181		0.7181		0.7192	

Comparing model X and Y we see the constant return level *ONE* is dominated by *NCYTM*. *NCYTM* turns out highly significant in model Y indicating as expected that the initial yield level derived from the term structure is an important determinant of HPR.

Striking results are obtained for the two duration measures. As expected *NCVOLA* turns out to be highly significant. The coefficient of *NCVOLA* measures the part of HPR which can be explained by duration and the actual shift in the non-callable term structure. A 20 basis point decrease in yield for a non-callable bond with a duration of say 4 would add an expected 8.8% to annualized HPR for the four week period.

The unexpected result is that *MBBVOLA* explains HPR for MBBs with even more precision. This implies that MBB durations computed from our model may be as useful for the prediction of returns, calculation of hedge ratios etc. as non-callable duration measures. The regression shows that a 20 basis point yield decrease would add an expected 6.8% to HPR for the MBB with MBB-duration of 4. The bias relative to non-callable duration means that the market adjusts MBB-prices less than expected following a change in non-callable yields. Some of the bias may be caused by the negative convexity shown in section 6.4. An other source of bias could be that our estimated prepayment model reacts too slowly on sudden changes in interest rates. However, the high precision of the estimate indicates that one could simply divide MBB durations by 1.3 for comparison. Later refinements of the model may remove the need for such adjustments.

Another interesting result is that net present value does seem to have a significant effect on HPR. Buying a MBB with estimated price one percent above market price would on average have increased annualized HPR more than 2%. No significant difference is found between MBBs and non-callable bonds[117].

Regression model Z uses dummy variables for 10%, 11% and 12% bonds. The results show that average return from these bonds have been below the yield that would have been expected from their scheduled payments. This is not an indication of below average performance because the contribution to HPR from average *MBBNPV* must be added[118]. 10% MBBs have an average *MBBNPV* of 0.56012 and the average contribution from *MBBNPV* and *MBB10* equals $-0.96083 + 2.50164 \cdot 0.56012 = 0.4404.$ Aver-

[117] These results must be interpreted with care because we use non-callable yield as well as net present value in the same regression. One could argue, that net present value is caused by random fluctuations in long term estimates, which would imply a relatively low promised yield when NPV is positive. However, the correlation between *MBBNPV* and *NCYTM* is small at 0.023. Correlation between *NCNPV* and *NCYTM* is -0.038. Further research with other specifications will hopefully clarify this issue.

[118] The average values of MBBNPV for different coupon rates have been calculated from the basic data-set. Figure 7.8 shows MBBNPV averaged accross coupon rates and closing year.

age *MBBNPV* for the 11% and 12% MBBs are at 1.4646 and 0.9235 respectively giving these bonds an average HPR lying 2.039 and 1.259 above bonds with similar risk and non-callable yields. Active traders could of course have done much better by selling at low NPV and buying when NPVs are high.

It would be very easy to improve the regression model using a more sophisticated correction of interest rate risk as well as a better measure of promised yield. We hope to follow up on these issues in forthcoming papers. The sample size could be extended to say 20.000 observations which would allow the test of almost any hypothesis on the relative pricing of bonds.

To summarize we have presented a new data-set of four-weekly holding period returns for the period 1988 to 1992. Plots of average return against standard deviation showed that the return from the MBB followed the prediction of the model quite closely. These findings were confirmed by a regression model on the full data set. The regression procedure allowed us to adjust for the differences in interest rate risk as well as the difference in non-callable yield. It was shown that average HPR for the 10%, 12% and especially 11% MBBs has been above HPR for bonds of similar risk and non-callable yield.

It was furthermore shown that the duration measure derived from the MBB pricing model has a precision which is equivalent to the duration measures for non-callable bonds although a simple adjustment must be used to compare the two. Finally the estimation indicated that active trading based on net present value estimates from the bond pricing models could contribute considerably to increased performance from MBBs as well as non-callable bonds.

7.6 Conclusion

This chapter represents the first attempt to base a pricing model for Danish mortgage backed bonds on actual prepayment experience. The analysis has required a considerable amount of data collection and program development much of which has been finished close to the deadline of this thesis. The work is therefore of a preliminary nature but the power of the empirical approach has hopefully been demonstrated. A prepayment function estimated on actual prepayment experience provides a common basis for the analysis of callable mortgage backed bonds, and the models can be compared on their predictive ability by use of prepayment data as well as actual market prices and holding period returns.

The particular model analysed in chapter 6 and tested in the current chapter has shown to be a surprisingly good description of the risk and return characteristics at least for the present sample of bonds. More sophisticated models can be designed as more data becomes available, but even at the present state of modelling we find it fair to conclude that the uncertainty regarding the relative pricing of Danish mortgage backed bonds has been reduced to a manageable level.

8 Dansk resumé (Danish Summary)

Formålet med denne afhandling er at udvikle og teste en model til prisfastsættelse af danske konverterbare realkreditobligationer. Konverterbare realkreditobligationer udgør mere end 52% af det samlede danske obligationsmarked med en cirkulerende beholdning på i alt 652 milliarder DKK. Hver enkelt konverterbare obligation udstedes med sikkerhed i tusindvis af individuelle huslån og betalingerne fra den konverterbare obligation er summen af de individuelle låntageres betalinger.

Prisfastsættelsen af konverterbare obligationer kompliceres af konverteringsretten, som tillader hver enkelt låntager at indfri det eksisterende lån til kurs 100 med efterfølgende optagelse af et nyt lån til gældende markedsrente. Konverteringsretten kan udnyttes på et vilkårligt tidspunkt i lånets løbetid. Da låntagerne normalt indfrier højt forrentede lån på tidspunkter, hvor markedsrenten er lav, vil konverteringsretten få stor betydning for afkastet ved investering i konverterbare obligationer.

Table 1.5: *Obligationer udstedt på Københavns Fondsbørs, Maj 1992*[119].

Obligationstype	Nominel værdi mio. DKK	Antal serier	Nominel værdi (%)	Antal serier (%)
Inkonverterbare obligationer	393.979	359	31,76	16,95
Konverterbare realkreditobligationer	651.578	1.713	52,53	80,88
Indeksobligationer	121.421	11	9,79	0,52
Variabel forrentede*	73.502	35	5,93	1,65
I alt	1.240.480	2.118	100,00	100,00

[119] Kilde: Tallene er hentet fra Københavns Fondsbørs officielle database 7. maj 1992. Obligationer i udenlandsk valuta og præmieobligationer er udelukket. *) Inkluderer 14 variabelt forrentede konverterbare realkreditobligationer med en samlet udestående nominel beholdning på 2095 millioner DKK.

Prisfastsættelsesmodellen bygger på en såkaldt arbitrage-fri rentestrukturmodel, i hvilken den aktuelle nulkuponrentestruktur kombineret med et estimat på den fremtidige rentevolatilitet fastlægger den mulige fremtidige stokastiske udvikling af rentestrukturen. I denne type model prisfastsættes obligationer med renteafhængigt betalingsforløb relativt til de aktuelle markedspriser for inkonverterbare obligationer med faste betalingsforløb. I afhandlingen udvides de arbitrage-frie modeller til brug i en efter-skat analyse.

Kapitel 2 gennemgår estimation af nulkuponrentestrukturer på det danske obligationsmarked ved brug af ugentlige data for perioden januar 1985 til juli 1992. Det vises, hvorledes estimationsteknikken må ændres i løbet af perioden som følge af manglen på lange inkonverterbare obligationer. Resultaterne peger desuden på en stigende markedseffektivitet for statsobligationer fra og med 1988.

Kapitel 3 gennemgår de arbitrage-frie rentestrukturmodeller, og der præsenteres en generel model til prisfastsættelse af konverterbare realkreditobligationer. Modellen afhænger af obligationens karakteristika, den stokastiske udvikling i rentestrukturen og en såkaldt konverteringsfunktion. Konverteringsfunktionen fastlægger hvordan konverteringsraten - dvs. andelen af låntagere som indfrier deres lån i en given periode - afhænger af grundlæggende parametre såsom renteniveau, restløbetid etc. Ved forskellige valg af konverteringsfunktion fremkommer forskellige prisfastsættelsemodeller.

En stor del af den danske litteratur omkring konverterbare obligationer har beskæftiget sig med skattesystemets indflydelse på låntagers konverteringsadfærd. Kapitel 4 indeholder en diskussion af arbitrageligevægt på obligationsmarkeder, hvor de enkelte investorer er underlagt forskellige skattesystemer. Som vist af en række forfattere vil der i et perfekt obligationsmarked opstå muligheder for ubegrænset skattearbitrage, hvis institutionelle investorer, der er fuldt beskattet af såvel kursgevinst som rente, frit kan handle med private investorer, som kun beskattes af renteindtægter. I kapitlet opstilles en model med institutionelle begrænsninger, som bevirker, at der ikke opstår ubegrænset skattearbitrage til trods for fri adgang til investering og låntagning i obligationer af enhver løbetid. De foreslåede begrænsninger kan ses som en skærpet udgave af det nuværende danske mindsterentesystem. Den institutionelle struktur indarbejdes i den arbitrage-frie rentestrukturmodel på en måde, der muliggør en beregningsmæssig effektiv prisfastsættelse af optioner såvel før som efter skat. Denne resulterende model benyttes ved samtlige efter-skat analyser i det følgende.

De resterende kapitler omhandler specifikation og estimation af konverteringsfunktionen. Kapitel 5 analyserer optimal konvertering. Konverteringsretten ses her som en amerikansk call-option på et underliggende inkonverterbart lån. Med udgangspunkt i teorien for amerikanske optioner bestemmes det kritiske renteniveau, ved hvilket den rationelle låntager udnytter sin konverteringsret. Kapitlet gennemgår, hvordan det kritiske renteniveau afhænger af løbetid, kuponrente, konverteringsomkostninger, låntagers skattesats samt de specielle aspekter omkring det danske kontantlånssystem.

I en separat analyse udvides modellen til at omfatte *ombytningsretten*, dvs. muligheden for at omlægge til lån med en højere kuponrente. Det vises, at ombytningsrettens værdi skyldes at markedets investorer er underlagt forskellige skattesystemer. Der opstilles en dynamisk model for optimal gældspleje i hvilken den rationelle låntager løbende fastlægger den optimale udnyttelse af såvel konverterings- som ombytningsret. Det vises, at ombytningsretten i denne model har en svag indirekte effekt på de konverterbare obligationers markedskurser, via ombytningsrettens påvirkning af låntagernes konverteringsadfærd.

Kapitel 5 afslutter med at vise, at den simple amerikanske optionsmodel, hvor alle låntagere konverterer på samme tidspunkt, medfører diskontinuiteter i de teoretiske kurser, som gør modellen uegnet som basis for en egentlig prisfastsættelsesmodel for konverterbare obligationer.

En empirisk anvendelig model for konverterbare obligationer må tage hensyn til individuelle forskelle mellem de enkelte låntagere og til at konverteringsadfærden bestemmes af mange faktorer, der ikke alle kan beskrives ved økonomisk rationel adfærd. I kapitel 6 udvikles en model, hvor den individuelle låntager konverterer, så snart et givet gevinstkrav er opnået. Gevinstkravet antages normaltfordelt på tværs af seriens låntagere. Ved at variere gevinstkravets fordeling kan der tages hensyn til vidt forskellig adfærd fra de enkelte låntagere.

Betalingerne fra den konverterbare obligation findes som følger. På et givet tidspunkt beregnes den gevinst, som kan opnåes med den gældende rentestruktur. De låntagere, for hvem den aktuelle gevinst overstiger den krævede gevinst, konverterer deres lån, mens resten betaler den normale ydelse. Konverteringsraten bestemmes på ud fra middelværdi og spredning i gevinstkravsfordelingen. I modsætning til optionsmodellen fra kapitel 5, hvor konverteringsraten enten var 0 eller 100%, er der i gevinstkrav-modellen et bredt spektrum af konverteringsrater fra nul til 100%.

Kapitlet gennemgår forskellige motiver til konvertering og diskuterer, hvordan de kan indbygges i gevinstkrav-modellen. De teoretiske prisers afhængighed af renteniveau, volatilitet, rentestrukturens form, restløbetiden, gevinstkravsfordelingen og skattesatsen analyseres og resultaterne sammenholdes med modellen fra kapitel 5. Det ses, at de vigtigste resultater fra analysen af optimal konverteringsadfærd kan indarbejdes ved at lade gevinstkravfordelingen afhænge af lånets restløbetid og eventuelt andre parametre.

Konverterbare obligationer har specielle renterisikoegenskaber, som følge af at betalingsforløbet bestemmes fra låntagersiden. Kapitel 6 fortsætter derfor med en diskussion af korrigerede varigheds- og konveksitetsmål for konverterbare obligationer. Det vises, at såkaldt negativ konveksitet bevirker, at afkastet af korrekt prisfastsatte konverterbare obligationer ligger under afkastet fra inkonverterbare obligationer med samme varighed pånær ved små ændringer i renten. Kapitlet afsluttes med et lille empirisk eksempel, som belyser, hvordan forskelle i konverteringsraterne mellem sammenlignelige obligationsserier udstedt af de tre store realkreditinstitutioner kan forklares med delvist uobserverede forskelle i låntagersammensætningen. Eksemplet viser desuden, at gevinstkravs-modellen har et vist empirisk potentiale.

Kapitel 7 opstiller, estimerer og aftester en komplet prisfastsættelsesmodel for danske konverterbare realkreditobligationer. Modellen er en variant af gevinstkravsmodellen fra kapitel 6 med den vigtige ændring, at låntageradfærden estimeres på basis af observerede udtrækningsprocenter. Herved indrages den tilgængelige viden, om hvordan låntagere historisk har konverteret ved forskellige renteniveauer, i vurderingen af de fremtidige konverteringsrater for de enkelte serier. I modellen, der så vidt vides er den første af sin art for det danske marked, anvendes et nykonstrueret datasæt bestående af udtrækningsprocenter for perioden 1988-92. Lignende empirisk baserede modeller er i de seneste år blevet udviklet på det amerikanske realkreditmarked.

Kapitel 7 starter med en gennemgang af de vigtigste amerikanske modeller. De observerede udtrækningsprocenter forklares i disse modeller ved såvel det hidtidige renteforløb som ved individuelle karakteristika for de enkelte serier. Grundet kontraktmæssige forskelle mellem det danske og det amerikanske marked kan modellerne ikke overtages direkte, men der argumenteres for et nyt sæt af forklarende variable, som tager højde for de specielle egenskaber ved det danske realkreditsystem.

Efter en kort gennemgang af datamaterialet opstilles den statistiske model til estimation af konverteringsfunktionen. I modsætning til de amerikanske modeller er gevinstkrav-modellen baseret på en "mikro-økonomisk" beslutningsmodel for de enkelte låntagere og det vises, at modellens parametre kan estimeres som en variant af de såkaldte probit-modeller. Forskellige specifikationer undersøges og det konkluderes, at en simpel model med fire parametre giver en rimelig beskrivelse af de historiske udtrækningsprocenter på det danske marked. Mere sofistikerede specifikationer bør testes efterhånden som datamængden øges. Den empiriske analyse identificerer desuden en gruppe af høj-risiko obligationer, som formentlig er kendetegnet ved en relativ stor andel af virksomhedslån.

Den estimerede konverteringsfunktion giver en bekvem opsummering af danske låntageres reaktion på det skiftende renteniveau. Konverteringsfunktionen indbygges herefter i den generelle arbitrage-frie prisfastsættelsesmodel sammen med estimater på nulkuponrentestrukturer og rentevolatilitet. Herfra udledes teoretiske priser og varigheder for en stikprøve bestående af 33 Nykredit-obligationer i perioden januar 1988 til juli 1992. Resultaterne viser, at de konverterbare obligationer har været relativt billige i store dele af perioden, selv om den generelle overenstemmelse mellem markedskurser og teoretiske kurser er pæn. Vi vil derfor forvente, at de konverterbare obligationer har givet et relativt højt afkast i perioden sammenlignet med inkonverterbare obligationer med samme rentefølsomhed.

Hypotesen testes på et datasæt bestående af fire-ugers afkast for såvel de 33 konverterbare obligationer som for samtlige store inkonverterbare statsobligationer i perioden 1988-92. Årlige plots af gennemsnitligt afkast mod standardafvigelse viser, at de konverterbare obligationers afkast følger prisfastsættelsesmodellens forudsigelser ganske tæt. Den grafiske analyse bekræftes af en foreløbig regressionsmodel på det fulde datasæt. Regressionsanalysen muliggør en mere præcis justering for forskelle i såvel renterisko som forskel i renten på de underliggende inkonverterbare obligationer. Resultaterne er foreløbige, men beregningerne tyder på, at det gennemsnitlige afkast for 10%, 12% og specielt 11%'s konverterbare obligationer har ligget over inkonverterbare obligationer med samme renterisko.

Det vises yderligere, at de korrigerede varighedsmål for konverterbare obligationer forklarer afkastudsving, som følge af ændringer i den inkonverterbare rentestruktur, med en præcision, som er sammenlignelig med normale varighedsmål for inkonverterbare obligationer. Der er dog nødvendigt at foretage en simpel justering for at sammenligne varigheder på tværs af de to obligationstyper. Denne observation tyder på, at de korrigerede varighedsmål er velegnede til måling af renterisiko, hedging etc.

Regressionsanalysen indikerer endeligt, at over- og undervurderinger beregnet ud fra den teoretiske prisfastsættelsesmodel har en signifikant indflydelse på det efterfølgende afkast. En aktiv handel på basis af den estimerede model ville formentligt have bidraget til et forøget afkast fra de konverterbare obligationer. Tilsvarende resultater fås for inkonverterbare obligationer.

Afhandlingens endelige konklusion er, at den opstillede model giver en god beskrivelse af det danske marked for konverterbare realkreditobligationer. Modellen er i overenstemmelse med grundlæggende finansiel teori omkring arbitrage-fri prisfastsættelse og den er testet på tilgængelige data for udtræksprocenter, markedskurser og afkast. I modsætning til tidligere modeller giver den empiriske tilgang et fælles grundlag for diskussion, test og brug af modellen. Efterhånden som estimationsteknikkerne videreudvikles og flere data bliver tilgængelige, vil de konverterbare obligationer forhåbentlig kunne prisfastsættes rutinemæssigt med næsten samme præcision som tilsvarende inkonverterbare obligationer.

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Appendix A Annualized 4-Weekly Returns 1988 - 1992

BondId	Coupon	Name	Last Year	Period	Avg. yield to maturity	Macauley Duration	No Peri-ods	Average repay-ment	Avg. Money market rate	Avg. % return	Std. dev. return	Risk adjusted excess return
0970271	10,00	3.S.	2017	1988	10,45	8,44	13	0,05	8,84	24,99	22,70	0,71
0970271	10,00	3.S.	2017	1989	10,86	8,14	13	0,12	9,88	5,44	15,18	-0,29
0970271	10,00	3.S.	2017	1991	10,35	7,99	13	0,11	10,14	17,53	14,00	0,53
0970271	10,00	3.S.	2017	1992	10,52	7,85	7	0,32	10,76	9,53	6,03	-0,20
0970433	9,00	2.S.	2006	1988	9,59	6,67	13	0,18	8,84	25,05	25,40	0,64
0970433	9,00	2.S.	2006	1989	10,27	6,26	13	0,20	9,88	2,90	18,06	-0,39
0970433	9,00	2.S.	2006	1990	10,87	5,88	13	0,23	11,60	9,25	27,36	-0,09
0970433	9,00	2.S.	2006	1991	10,21	5,68	13	0,26	10,14	16,58	17,06	0,38
0970433	9,00	2.S.	2006	1992	10,08	5,52	7	0,39	10,76	6,41	9,79	-0,44
0970441	9,00	3.S.	2017	1988	9,81	8,58	13	0,06	8,84	30,03	28,61	0,74
0970441	9,00	3.S.	2017	1989	10,32	8,22	13	0,07	9,88	3,36	22,05	-0,30
0970484	9,00	20.S.	2006	1988	9,72	6,75	13	0,18	8,84	25,20	20,70	0,79
0970484	9,00	20.S.	2006	1989	10,21	6,38	13	0,20	9,88	3,27	20,74	-0,32
0970484	9,00	20.S.	2006	1990	10,90	5,97	13	0,23	11,60	9,37	22,33	-0,10
0970484	9,00	20.S.	2006	1991	9,88	5,83	13	0,26	10,14	14,78	14,50	0,32
0970484	9,00	20.S.	2006	1992	10,09	5,62	7	0,54	10,76	12,61	21,36	0,09
0970492	9,00	30.S.	2017	1988	9,84	8,69	13	0,07	8,84	30,26	29,07	0,74
0970492	9,00	30.S.	2017	1989	10,27	8,36	13	0,07	9,88	3,41	22,97	-0,28
0970492	9,00	30.S.	2017	1990	10,68	8,06	13	0,08	11,60	9,23	36,16	-0,07
0970492	9,00	30.S.	2017	1991	9,88	8,17	13	0,09	10,14	17,86	18,60	0,41
0970522	9,00	2.S.	2007	1988	9,70	6,82	13	0,17	8,84	25,46	24,45	0,68
0970522	9,00	2.S.	2007	1989	10,29	6,43	13	0,19	9,88	3,92	16,84	-0,35
0970522	9,00	2.S.	2007	1992	10,09	5,72	7	0,41	10,76	8,39	17,90	-0,13
0970859	8,00	2.S.	2006	1988	9,17	6,85	13	0,19	8,84	27,89	27,38	0,70
0970859	8,00	2.S.	2006	1989	10,11	6,36	13	0,22	9,88	2,17	19,70	-0,39
0970859	8,00	2.S.	2006	1990	10,70	5,98	13	0,24	11,60	7,48	23,00	-0,18
0970859	8,00	2.S.	2006	1991	9,58	5,86	13	0,28	10,14	17,02	15,16	0,45
0970859	8,00	2.S.	2006	1992	9,70	5,66	7	0,42	10,76	8,11	15,76	-0,17
0971316	10,00	2.S.	2007	1988	10,35	6,74	13	0,15	8,84	21,66	21,52	0,60
0971316	10,00	2.S.	2007	1989	10,91	6,40	13	0,18	9,88	6,24	12,75	-0,29
0971316	10,00	2.S.	2007	1990	11,16	6,08	13	0,18	11,60	9,76	21,87	-0,08
0971316	10,00	2.S.	2007	1991	10,34	5,78	13	0,36	10,14	15,19	12,82	0,39
0971316	10,00	2.S.	2007	1992	10,51	5,75	7	0,81	10,76	8,54	4,27	-0,52
0971413	11,00	3.S.	2017	1988	11,00	4,80	13	0,04	8,84	22,50	23,19	0,59
0971413	11,00	3.S.	2017	1989	11,54	7,89	13	1,05	9,88	7,86	7,53	-0,27
0971413	11,00	3.S.	2017	1992	11,34	7,60	7	2,34	10,76	8,33	5,36	-0,45
0971421	11,00	2.S.	2007	1988	10,41	1,69	13	0,31	8,84	18,56	19,21	0,51
0971421	11,00	2.S.	2007	1989	11,58	6,29	13	1,90	9,88	7,60	9,07	-0,25
0971421	11,00	2.S.	2007	1990	11,78	5,96	13	0,17	11,60	12,34	18,80	0,04
0971421	11,00	2.S.	2007	1991	11,05	5,57	13	1,33	10,14	13,08	6,21	0,47

Appendix A Annualized 4-Weekly Returns 1988 - 1992

BondId	Coupon	Name	Last Year	Period	Avg. yield to maturity	Macauley Duration	No Peri-ods	Average repay-ment	Avg. Money market rate	Avg. % return	Std. dev. return	Risk adjusted excess return
0971421	11,00	2.S.	2007	1992	11,25	5,69	7	2,82	10,76	8,89	6,91	-0,27
0971480	10,00	12.S.S	2007	1988	10,27	5,76	13	0,41	8,84	19,27	18,82	0,55
0971480	10,00	12.S.S	2007	1990	11,02	5,27	13	0,45	11,60	12,54	27,47	0,03
0971650	11,00	12.S.S	2010	1988	11,05	5,07	13	0,42	8,84	18,15	18,84	0,49
0971650	11,00	12.S.S	2010	1989	11,59	5,67	13	0,61	9,88	7,11	9,24	-0,30
0971650	11,00	12.S.S	2010	1990	11,69	5,45	13	0,43	11,60	12,62	16,98	0,06
0971650	11,00	12.S.S	2010	1991	11,00	5,36	13	0,90	10,14	13,93	5,42	0,70
0971650	11,00	12.S.S	2010	1992	11,26	5,39	7	1,44	10,76	9,41	6,66	-0,20
0971669	11,00	3.S	2020	1988	10,77	3,08	13	0,05	8,84	23,01	23,30	0,61
0971669	11,00	3.S	2020	1989	11,60	8,02	13	1,37	9,88	6,03	13,22	-0,29
0971669	11,00	3.S	2020	1990	11,94	7,79	13	0,04	11,60	12,20	16,54	0,04
0971669	11,00	3.S	2020	1991	11,20	7,10	13	1,12	10,14	14,55	8,30	0,53
0971669	11,00	3.S	2020	1992	11,36	7,79	7	2,07	10,76	9,56	5,14	-0,23
0971677	11,00	2.S	2010	1988	9,86	1,25	13	0,39	8,84	20,03	21,17	0,53
0971677	11,00	2.S	2010	1989	11,59	6,56	13	2,90	9,88	5,97	11,76	-0,33
0971677	11,00	2.S	2010	1990	11,79	6,26	13	0,14	11,60	12,62	14,63	0,07
0971677	11,00	2.S	2010	1991	11,18	5,45	13	2,18	10,14	13,04	7,23	0,40
0971677	11,00	2.S	2010	1992	11,33	6,02	7	2,84	10,76	10,14	3,29	-0,19
0971685	10,00	13.S.S	2020	1988	10,46	7,03	13	0,26	8,84	22,69	19,97	0,69
0971685	10,00	13.S.S	2020	1989	11,02	6,91	13	0,29	9,88	5,34	12,09	-0,38
0971693	10,00	12.S.S	2010	1988	10,39	6,05	13	0,41	8,84	20,11	17,75	0,64
0971693	10,00	12.S.S	2010	1989	11,00	5,84	13	0,42	9,88	5,30	13,16	-0,35
0971693	10,00	12.S.S	2010	1990	11,34	5,65	13	0,42	11,60	11,06	19,18	-0,03
0971693	10,00	12.S.S	2010	1991	10,41	5,55	13	0,62	10,14	15,03	9,14	0,54
0971693	10,00	12.S.S	2010	1992	10,44	5,57	7	1,12	10,76	9,50	4,77	-0,26
0971707	10,00	3.S.	2020	1988	10,44	8,68	13	0,05	8,84	25,59	26,91	0,62
0971707	10,00	3.S.	2020	1989	10,96	8,34	13	0,07	9,88	5,07	14,90	-0,32
0971707	10,00	3.S.	2020	1990	11,31	8,08	13	0,05	11,60	9,37	23,92	-0,09
0971707	10,00	3.S.	2020	1991	10,44	8,25	13	0,11	10,14	18,24	15,12	0,54
0971707	10,00	3.S.	2020	1992	10,52	8,16	7	0,25	10,76	8,46	3,90	-0,59
0971715	10,00	2.S.	2010	1988	10,38	7,12	13	0,13	8,84	22,71	22,94	0,60
0971715	10,00	2.S.	2010	1989	10,94	6,75	13	0,14	9,88	5,31	14,57	-0,31
0971715	10,00	2.S.	2010	1990	11,29	6,47	13	0,15	11,60	9,96	22,76	-0,07
0971715	10,00	2.S.	2010	1991	10,42	6,31	13	0,29	10,14	16,41	12,38	0,51
0971715	10,00	2.S.	2010	1992	10,58	6,22	7	0,61	10,76	8,76	6,38	-0,31
0971774	12,00	3.S.	2020	1988	9,39	0,90	13	1,17	8,84	19,73	16,06	0,68
0971774	12,00	3.S.	2020	1989	12,15	7,78	13	3,50	9,88	10,39	7,98	0,06
0971774	12,00	3.S.	2020	1990	12,45	6,44	13	0,60	11,60	10,16	18,90	-0,08
0971774	12,00	3.S.	2020	1991	11,97	5,72	13	6,17	10,14	11,06	7,40	0,12
0971774	12,00	3.S.	2020	1992	12,11	7,50	7	4,79	10,76	9,48	2,92	-0,44

Appendix A Annualized 4-Weekly Returns 1988 - 1992

BondId	Coupon	Name	Last Year	Period	Avg. yield to maturity	Macauley Duration	No Peri-ods	Average repay-ment	Avg. Money market rate	Avg. % return	Std. dev. return	Risk adjusted excess return
0971782	12,00	3.S	2017	1988	9,79	1,23	13	0,90	8,84	18,65	15,03	0,65
0971782	12,00	3.S	2017	1989	12,15	7,70	13	2,96	9,88	9,86	8,02	0,00
0971790	12,00	2.S.	2010	1988	9,52	0,92	13	3,23	8,84	16,04	13,69	0,53
0971790	12,00	2.S.	2010	1989	12,10	6,43	13	3,09	9,88	10,41	7,71	0,07
0971790	12,00	2.S.	2010	1990	12,35	3,51	13	2,05	11,60	10,80	15,86	-0,05
0971790	12,00	2.S.	2010	1991	11,88	4,86	13	7,26	10,14	10,59	6,50	0,07
0971790	12,00	2.S.	2010	1992	12,03	5,88	7	3,13	10,76	10,32	6,00	-0,07
0971804	12,00	2.S.	2007	1988	8,72	0,84	13	3,90	8,84	14,96	12,37	0,49
0971804	12,00	2.S.	2007	1989	12,01	6,25	13	3,53	9,88	10,03	7,16	0,02
0971804	12,00	2.S.	2007	1992	11,91	5,64	7	2,30	10,76	8,46	3,76	-0,61
0971936	9,00	2.S.	2010	1989	10,45	7,01	13	0,15	9,88	2,56	18,32	-0,40
0971936	9,00	2.S.	2010	1990	11,04	6,66	13	0,16	11,60	8,92	28,22	-0,10
0971936	9,00	2.S.	2010	1991	9,96	6,65	13	0,17	10,14	18,18	16,05	0,50
0971936	9,00	2.S.	2010	1992	10,10	6,46	7	0,26	10,76	6,46	11,15	-0,39
0971944	9,00	3.S.	2020	1989	10,41	8,65	13	0,06	9,88	2,87	20,31	-0,35
0971944	9,00	3.S.	2020	1990	10,94	8,30	13	0,05	11,60	8,25	33,74	-0,10
0971944	9,00	3.S.	2020	1991	9,92	8,60	13	0,06	10,14	19,83	19,16	0,51
0971944	9,00	3.S.	2020	1992	10,00	8,46	7	0,09	10,76	6,24	11,22	-0,40
0971979	9,00	12.S.S.	2010	1989	10,46	6,01	13	0,40	9,88	3,49	15,14	-0,42
0971979	9,00	12.S.S.	2010	1990	11,08	5,77	13	0,41	11,60	10,33	24,37	-0,05
0971979	9,00	12.S.S.	2010	1991	9,89	5,84	13	0,42	10,14	16,10	10,29	0,58
0971979	9,00	12.S.S.	2010	1992	9,76	5,75	7	0,61	10,76	8,42	6,84	-0,34
0971987	9,00	13.S.S.	2020	1989	10,50	7,03	13	0,26	9,88	3,23	17,79	-0,37
0971987	9,00	13.S.S.	2020	1990	11,05	6,82	13	0,27	11,60	8,44	30,31	-0,10
0971987	9,00	13.S.S.	2020	1991	10,02	7,05	13	0,27	10,14	18,44	17,69	0,47
0973203	9,00	2A.S.	2012	1991	10,01	7,10	13	0,16	10,14	19,20	17,58	0,52
0973203	9,00	2A.S.	2012	1992	10,14	6,93	7	0,23	10,76	5,49	10,15	-0,52
0973211	9,00	12A.S.S.	2012	1991	10,01	6,12	13	0,42	10,14	17,74	14,71	0,52
0973211	9,00	12A.S.S.	2012	1992	10,16	6,00	7	0,58	10,76	5,94	9,17	-0,52
0973238	9,00	3A.S.	2022	1991	9,96	8,83	13	0,06	10,14	21,05	20,68	0,53
0973238	9,00	3A.S.	2022	1992	10,12	8,66	7	0,08	10,76	4,40	12,72	-0,50
0973246	9,00	13A.S.S.	2022	1991	9,98	7,32	13	0,28	10,14	19,40	16,50	0,56
0973246	9,00	13A.S.S.	2022	1992	10,13	7,20	7	0,38	10,76	5,17	10,54	-0,53
0973254	10,00	2A.S.	2012	1991	10,47	6,96	13	0,16	10,14	17,10	13,42	0,52
0973254	10,00	2A.S.	2012	1992	10,68	6,81	7	0,29	10,76	7,94	5,92	-0,48
0973262	10,00	12A.S.S	2012	1991	10,42	5,92	13	0,44	10,14	16,52	10,07	0,63
0973262	10,00	12A.S.S	2012	1992	10,57	5,86	7	0,73	10,76	8,43	3,96	-0,59
0973270	10,00	3A.S.	2022	1991	10,45	8,59	13	0,08	10,14	18,82	14,27	0,61
0973270	10,00	3A.S.	2022	1992	10,62	8,43	7	0,12	10,76	7,47	5,57	-0,59
0973289	10,00	13A.S.S.	2022	1991	10,41	7,10	13	0,29	10,14	17,24	10,95	0,65

Appendix A Annualized 4-Weekly Returns 1988 - 1992

BondId	Coupon	Name	Last Year	Period	Avg. yield to maturity	Macauley Duration	No Peri-ods	Average repay-ment	Avg. Money market rate	Avg. % return	Std. dev. return	Risk adjusted excess return
0973289	10,00	13A.S.S.	2022	1992	10,55	7,00	7	0,45	10,76	7,99	4,11	-0,67
0973297	11,00	2A.S.	2012	1991	11,31	4,95	13	7,77	10,14	12,84	8,47	0,32
0973297	11,00	2A.S.	2012	1992	11,46	6,63	7	3,57	10,76	10,41	6,93	-0,05
0990493	12,00	S 2001	2001	1988	9,16	4,48	13	0,59	8,84	19,97	16,05	0,69
0990493	12,00	S 2001	2001	1989	10,36	4,15	13	0,64	9,88	3,16	14,47	-0,46
0990493	12,00	S 2001	2001	1990	10,72	3,89	13	0,70	11,60	10,15	20,28	-0,07
0990493	12,00	S 2001	2001	1991	9,66	3,72	13	0,77	10,14	15,35	11,46	0,45
0990493	12,00	S 2001	2001	1992	9,84	3,57	7	0,00	10,76	6,25	13,93	-0,32
0990701	10,00	S 1993	1993	1988	8,61	2,03	13	1,28	8,84	12,90	9,86	0,41
0990701	10,00	S 1993	1993	1989	10,73	1,62	13	1,54	9,88	4,33	7,89	-0,70
0990701	10,00	S 1993	1993	1990	10,59	1,23	13	1,92	11,60	11,06	9,10	-0,06
0990701	10,00	S 1993	1993	1991	9,86	0,80	13	2,56	10,14	9,77	3,16	-0,12
0990701	10,00	S 1993	1993	1992	10,44	0,81	7	7,14	10,76	7,20	3,94	-0,90
0990728	10,00	St.lån	1993	1988	8,88	3,70	13	0,00	8,84	16,79	16,80	0,47
0990728	10,00	St.lån	1993	1989	10,43	2,99	13	0,00	9,88	3,77	10,98	-0,56
0990728	10,00	St.lån	1993	1990	10,44	2,25	13	0,00	11,60	10,96	11,70	-0,05
0990728	10,00	St.lån	1993	1991	9,77	1,43	13	0,00	10,14	11,06	4,73	0,19
0990728	10,00	St.lån	1993	1992	10,50	0,98	7	0,00	10,76	8,25	1,93	-1,30
0990736	10,00	S 1999	1999	1988	9,12	4,17	13	0,64	8,84	17,60	15,11	0,58
0990736	10,00	S 1999	1999	1989	10,37	3,81	13	0,70	9,88	3,02	13,57	-0,51
0990736	10,00	S 1999	1999	1990	10,60	3,53	13	0,77	11,60	10,50	18,59	-0,06
0990736	10,00	S 1999	1999	1991	9,69	3,28	13	0,85	10,14	14,27	10,04	0,41
0990736	10,00	S 1999	1999	1992	9,96	3,41	7	1,79	10,76	6,37	12,06	-0,36
0990744	10,00	S 2004	2004	1988	9,30	5,57	13	0,45	8,84	21,67	17,97	0,71
0990744	10,00	S 2004	2004	1989	10,18	5,24	13	0,48	9,88	3,10	15,58	-0,44
0990744	10,00	S 2004	2004	1990	10,63	4,99	13	0,51	11,60	9,30	25,31	-0,09
0990744	10,00	S 2004	2004	1991	9,52	4,89	13	0,55	10,14	16,63	13,17	0,49
0990744	10,00	S 2004	2004	1992	9,68	4,34	7	1,10	10,76	6,56	13,71	-0,31
0990825	10,00	S 1994	1994	1988	8,72	2,39	13	1,10	8,84	13,51	11,15	0,42
0990825	10,00	S 1994	1994	1989	10,57	1,99	13	1,28	9,88	4,11	8,61	-0,67
0990825	10,00	S 1994	1994	1990	10,60	1,62	13	1,54	11,60	10,74	10,19	-0,08
0990825	10,00	S 1994	1994	1991	9,78	1,23	13	1,92	10,14	10,56	3,46	0,12
0990825	10,00	S 1994	1994	1992	10,15	1,27	7	4,76	10,76	7,54	2,53	-1,27
0990884	10,00	S 1990	1990	1988	7,14	0,55	13	2,56	8,84	10,87	4,83	0,42
0990892	10,00	S 1995	1995	1988	8,86	2,73	13	0,96	8,84	14,16	11,34	0,47
0990892	10,00	S 1995	1995	1989	10,51	2,35	13	1,10	9,88	4,26	8,44	-0,67
0990892	10,00	S 1995	1995	1990	10,62	2,00	13	1,28	11,60	10,07	11,22	-0,14
0990892	10,00	S 1995	1995	1991	9,65	1,64	13	1,54	10,14	11,54	5,32	0,26
0990892	10,00	S 1995	1995	1992	9,99	1,70	7	3,57	10,76	7,28	3,53	-0,99
0990914	10,00	St.lån	1994	1988	9,02	4,08	13	0,00	8,84	17,29	23,22	0,36

Appendix A Annualized 4-Weekly Returns 1988 - 1992

BondId	Coupon	Name	Last Year	Period	Avg. yield to maturity	Macauley Duration	No Peri-ods	Average repay-ment	Avg. Money market rate	Avg. % return	Std. dev. return	Risk adjusted excess return
0990914	10,00	St.lån	1994	1989	10,33	3,43	13	0,00	9,88	3,90	11,68	-0,51
0990914	10,00	St.lån	1994	1990	10,38	2,75	13	0,00	11,60	11,09	15,66	-0,03
0990914	10,00	St.lån	1994	1991	9,57	2,00	13	0,00	10,14	12,11	7,00	0,28
0990914	10,00	St.lån	1994	1992	10,10	1,64	7	0,00	10,76	7,42	6,16	-0,54
0990922	10,00	St.lån	1989	1988	8,39	1,01	13	0,00	8,84	11,19	7,41	0,32
0990949	10,00	St.lån	1992	1988	8,70	3,17	13	0,00	8,84	14,46	13,37	0,42
0990949	10,00	St.lån	1992	1989	10,51	2,41	13	0,00	9,88	4,12	8,80	-0,65
0990949	10,00	St.lån	1992	1990	10,50	1,59	13	0,00	11,60	11,05	10,05	-0,05
0990949	10,00	St.lån	1992	1991	10,07	0,68	13	0,00	10,14	10,45	2,73	0,11
0990949	10,00	St.lån	1992	1992	12,02	0,15	7	0,00	10,76	9,28	1,58	-0,94
0990957	10,00	St.lån	1991	1988	8,60	2,17	13	0,00	8,84	12,92	9,67	0,42
0990957	10,00	St.lån	1991	1989	10,85	1,34	13	0,00	9,88	5,27	6,83	-0,67
0990957	10,00	St.lån	1991	1990	10,81	0,43	13	0,00	11,60	11,08	4,84	-0,11
0990965	10,00	St.lån	1990	1988	8,55	1,84	13	0,00	8,84	12,55	10,83	0,34
0990965	10,00	St.lån	1990	1989	11,03	0,93	13	0,00	9,88	6,21	5,51	-0,67
0991023	10,00	S 1991	1991	1988	8,60	1,42	13	2,56	8,84	10,54	6,44	0,26
0991112	9,00	S 1991	1991	1988	8,52	1,43	13	2,56	8,84	10,74	7,20	0,26
0991112	9,00	S 1991	1991	1989	10,74	1,01	13	3,85	9,88	6,60	4,91	-0,67
0991139	9,00	St.lån	1989	1988	10,12	1,01	13	0,00	8,84	11,06	7,72	0,29
0991147	9,00	St.lån	1990	1988	8,52	1,84	13	0,00	8,84	12,81	11,67	0,34
0991147	9,00	St.lån	1990	1989	10,81	0,93	13	0,00	9,88	6,20	4,99	-0,74
0991252	8,00	St.lån	1989	1988	9,68	1,85	13	0,00	8,84	10,52	7,64	0,22
0991260	8,00	St.lån	1990	1988	8,22	1,85	13	0,00	8,84	11,58	8,61	0,32
0991260	8,00	St.lån	1990	1989	9,92	0,93	13	0,00	9,88	5,79	5,23	-0,78
0991279	8,00	St.lån	1991	1988	8,29	2,21	13	0,00	8,84	13,28	11,96	0,37
0991279	8,00	St.lån	1991	1989	10,09	1,35	13	0,00	9,88	4,53	6,55	-0,82
0991279	8,00	St.lån	1991	1990	10,01	0,43	13	0,00	11,60	10,63	3,85	-0,25
0991287	8,00	St.lån	1992	1988	8,24	3,25	13	0,00	8,84	13,76	14,24	0,35
0991287	8,00	St.lån	1992	1989	9,98	2,46	13	0,00	9,88	3,87	8,89	-0,68
0991287	8,00	St.lån	1992	1990	10,18	1,61	13	0,00	11,60	10,01	10,81	-0,15
0991287	8,00	St.lån	1992	1991	9,46	0,68	13	0,00	10,14	10,12	4,92	0,00
0991287	8,00	St.lån	1992	1992	10,87	0,15	7	0,00	10,76	8,72	1,00	-2,04
0991295	8,00	S 1986/	1989	1988	9,56	1,39	13	3,85	8,84	8,78	5,48	-0,01
0991309	8,00	S 1992	1992	1988	8,23	1,85	13	1,92	8,84	9,77	7,76	0,12
0991309	8,00	S 1992	1992	1989	9,58	1,44	13	2,56	9,88	5,14	7,17	-0,66
0991309	8,00	S 1992	1992	1990	9,72	1,02	13	3,85	11,60	8,36	6,69	-0,48
0991368	10,00	Stgb.I	1989	1988	8,27	0,19	13	0,00	8,84	9,81	3,65	0,27
0991376	10,00	S 1987/	1990	1988	7,60	0,99	13	2,56	8,84	11,10	7,62	0,30
0991376	10,00	S 1987/	1990	1989	11,99	0,51	13	3,85	9,88	7,29	4,44	-0,58
0991384	10,00	Stgb.II	1989	1988	8,43	0,44	13	0,00	8,84	10,40	3,98	0,39

Appendix A Annualized 4-Weekly Returns 1988 - 1992

BondId	Coupon	Name	Last Year	Period	Avg. yield to maturity	Macauley Duration	No Peri-ods	Average repay-ment	Avg. Money market rate	Avg. % return	Std. dev. return	Risk adjusted excess return
0991392	10,00	Stgb.III	1989	1988	8,31	0,69	13	0,00	8,84	10,69	5,33	0,35
0991406	10,00	Stgb.IV	1989	1988	8,36	0,94	13	0,00	8,84	11,16	5,98	0,39
0991414	10,00	Stgb. I	1990	1988	8,38	1,10	13	0,00	8,84	11,52	6,96	0,38
0991414	10,00	Stgb. I	1990	1989	12,02	0,19	13	0,00	9,88	7,76	2,80	-0,76
0991422	10,00	St.lån	1996	1989	10,12	5,28	13	0,00	9,88	3,13	15,73	-0,43
0991422	10,00	St.lån	1996	1990	10,44	4,71	13	0,00	11,60	10,25	25,15	-0,05
0991422	10,00	St.lån	1996	1991	9,41	4,11	13	0,00	10,14	14,92	11,93	0,40
0991422	10,00	St.lån	1996	1992	9,84	3,57	7	0,00	10,76	5,96	16,22	-0,30
0991430	10,00	Stgb.II	1990	1989	11,27	0,44	13	0,00	9,88	7,35	3,23	-0,78
0991449	9,00	Stgb. III	1990	1989	10,90	0,69	13	0,00	9,88	6,67	3,71	-0,86
0991457	9,00	Stgb. II	1990	1989	11,04	0,44	13	0,00	9,88	7,35	3,19	-0,79
0991465	9,00	Stgb IV	1990	1989	10,91	0,94	13	0,00	9,88	6,29	4,66	-0,77
0991503	9,00	St.lån	1996	1989	10,13	5,37	13	0,00	9,88	1,31	19,72	-0,43
0991503	9,00	St.lån	1996	1990	10,55	4,78	13	0,00	11,60	10,67	24,80	-0,04
0991503	9,00	St.lån	1996	1991	9,29	4,17	13	0,00	10,14	16,26	12,62	0,48
0991503	9,00	St.lån	1996	1992	9,62	3,63	7	0,00	10,76	5,91	13,78	-0,35
0991511	9,00	St.lån	1994	1989	10,31	4,15	13	0,00	9,88	2,22	13,74	-0,56
0991511	9,00	St.lån	1994	1990	10,51	3,45	13	0,00	11,60	10,95	16,04	-0,04
0991511	9,00	St.lån	1994	1991	9,46	2,69	13	0,00	10,14	13,59	8,55	0,40
0991511	9,00	St.lån	1994	1992	9,88	2,15	7	0,00	10,76	6,79	8,91	-0,45
0991546	9,00	St.lån	1992	1989	10,43	2,43	13	0,00	9,88	4,48	8,61	-0,63
0991546	9,00	St.lån	1992	1990	10,57	1,60	13	0,00	11,60	10,28	8,51	-0,15
0991546	9,00	St.lån	1992	1991	9,89	0,68	13	0,00	10,14	10,56	3,07	0,14
0991546	9,00	St.lån	1992	1992	11,80	0,15	7	0,00	10,76	9,16	0,70	-2,30
0991554	9,00	St.lån	1998	1989	10,07	6,38	13	0,00	9,88	1,41	21,94	-0,39
0991554	9,00	St.lån	1998	1990	10,55	5,87	13	0,00	11,60	10,98	28,30	-0,02
0991554	9,00	St.lån	1998	1991	9,17	5,41	13	0,00	10,14	17,93	14,58	0,53
0991554	9,00	St.lån	1998	1992	9,38	4,87	7	0,00	10,76	4,61	13,63	-0,45
0991562	9,00	stgb. I	1991	1990	10,26	0,19	13	0,00	11,60	11,59	3,70	0,00
0991570	9,00	Stgb.II	1991	1990	10,78	0,44	13	0,00	11,60	11,22	4,01	-0,09
0991589	9,00	Stgb. III	1991	1990	10,65	0,69	13	0,00	11,60	11,26	5,77	-0,06
0991597	9,00	Stgb.IV	1991	1990	10,61	0,94	13	0,00	11,60	11,55	6,87	-0,01
0991600	9,00	Stgb. I	1992	1990	10,62	1,11	13	0,00	11,60	11,41	8,13	-0,02
0991600	9,00	Stgb. I	1992	1991	9,71	0,20	13	0,00	10,14	10,24	2,25	0,04
0991619	9,00	St.lån	2000	1990	10,51	6,77	13	0,00	11,60	9,32	31,24	-0,07
0991619	9,00	St.lån	2000	1991	9,03	6,47	13	0,00	10,14	19,81	18,14	0,53
0991619	9,00	St.lån	2000	1992	9,13	5,92	7	0,00	10,76	4,37	15,62	-0,41
0991627	9,00	STGB.II	1992	1991	9,83	0,45	13	0,00	10,14	10,49	2,87	0,12
0991635	9,00	STGB.III	1992	1991	9,87	0,70	13	0,00	10,14	10,56	3,49	0,12
0991635	9,00	STGB.III	1992	1992	11,21	0,16	7	0,00	10,76	9,41	0,47	-2,88

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BondId	Coupon	Name	Last Year	Period	Avg. yield to maturity	Macauley Duration	No Peri-ods	Average repay-ment	Avg. Money market rate	Avg. % return	Std. dev. return	Risk adjusted excess return
0991643	9,00	St.lån	1995	1991	9,40	3,46	13	0,00	10,14	15,01	10,66	0,46
0991643	9,00	St.lån	1995	1992	9,70	2,92	7	0,00	10,76	5,88	9,63	-0,51
0991651	9,00	STGB. IV	1992	1991	9,73	0,95	13	0,00	10,14	10,83	4,35	0,16
0991651	9,00	STGB. IV	1992	1992	10,93	0,41	7	0,00	10,76	8,97	1,24	-1,44
0991678	9,00	STGB. I	1993	1991	9,86	1,11	13	0,00	10,14	11,07	4,68	0,20
0991678	9,00	STGB. I	1993	1992	10,64	0,66	7	0,00	10,76	8,43	1,48	-1,57
0991686	9,00	STGB. II	1993	1992	10,47	0,91	7	0,00	10,76	8,21	2,32	-1,10
0991694	8,00	STGB III	1993	1992	10,15	1,09	7	0,00	10,76	7,51	4,64	-0,70
0991708	8,00	STGB. IV	1993	1992	10,07	1,34	7	0,00	10,76	7,26	5,14	-0,68
0991716	8,00	St.lån	2003	1992	9,02	7,48	7	0,00	10,76	3,26	20,00	-0,38
0991724	8,00	STGB. I	1994	1992	10,03	1,59	7	0,00	10,76	7,24	6,79	-0,52